Kruzhkov's uniqueness theorem: (Step 4)

Any weak solution u of the problem

$$\begin{cases} u_t + f(u)_x = 0, & x \in \mathbb{R}^1, t > 0 \\ u(x,0) = u_0(x) \end{cases}$$

which satisfy the Kruzkov entropy condition is unique.

Proof: Step 1: Let u and $\bar{u}$ are two solutions. Then

$$(1) \quad \int_0^\infty \int |u-\alpha| v_t + \operatorname{sgn}(u-\alpha)(f(u)-f(\alpha)) v_x \, dx\, dt \geq 0$$
$$\forall \alpha, \forall v \in C_c^\infty(\mathbb{R}^1 \times (0,\infty)), v \geq 0$$

$$(2) \quad \int_0^\infty \int |\bar{u}-\bar{\alpha}| \bar{v}_s + \operatorname{sgn}(\bar{u}-\bar{\alpha})(f(\bar{u})-f(\bar{\alpha})) \bar{v}_y \, dy\, ds \geq 0$$
$$\forall \bar{\alpha}, \forall \bar{v} \in C_c^\infty(\mathbb{R}^1 \times (0,\infty)), \bar{v} \geq 0.$$

Now take $w = C_c^\infty(R \times R \times (0,\infty) \times (0,\infty))$, $w \geq 0$, $w(x,y,t,s)$ in (1) and Fix (y,s) and take $\alpha = \bar{u}(y,s)$, $v(x,t) = w(x,y,t,s)$ in (2), we fix (x,t) and take $\bar{\alpha} = u(x,t)$, $\bar{v}(y,s) = w(x,y,t,s)$ get after integration,

$$(1)' \quad \int_0^\infty\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty |u(x,t)-\bar{u}(y,s)| w_t + \operatorname{sgn}(u(x,t)-\bar{u}(y,s))\big(f(u(x,t))-f(\bar{u}(y,s))\big) w_x \, dx\,dy\,dt\,ds \geq 0$$

$$(2)' \quad \int_0^\infty\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty |\bar{u}(y,s)-u(x,t)| w_s + \operatorname{sgn}(\bar{u}(y,s)-u(x,t))\big(f(\bar{u}(y,s))-f(u(x,t))\big) w_y \, dx\,dy\,dt\,ds \geq 0$$

Adding $(1)'$ and $(2)'$ we get

$$3). \quad \int_0^\infty\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty |u(x,t)-\bar{u}(y,s)|(w_t+w_s) + \operatorname{sgn}(u(x,t)-\bar{u}(y,s))\big(f(u(x,t))-f(\bar{u}(y,s))\big)(w_x+w_y)\, dx\,dy\,dt\,ds \geq 0$$

Step 2: (Choice of test functions $w(x,y,t,s)$): As usual let η_ε is the Friedrich's mollifier.

$$w(x,y,t,s) = \eta_\varepsilon\left(\frac{x-y}{2}\right)\cdot \eta_\varepsilon\left(\frac{t-s}{2}\right)\cdot \varphi\left(\frac{x+y}{2}, \frac{t+s}{2}\right)$$

where $\varphi \in C_c^\infty(R \times (0,\infty))$, $\varphi \geq 0$. Put this choice of w in (3) we get

$$(4) \quad \int_0^\infty\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty \Big\{ |u(x,t)-\bar{u}(y,s)| \varphi_t\left(\frac{x+y}{2}, \frac{t+s}{2}\right) + \operatorname{sgn}(u(x,t)-\bar{u}(y,s))\big(f(u(x,t))-f(\bar{u}(y,s))\big)\cdot \varphi_x\left(\frac{x+y}{2}, \frac{t+s}{2}\right) \Big\}$$
$$\eta_\varepsilon\left(\frac{x-y}{2}\right)\eta_\varepsilon\left(\frac{t-s}{2}\right) dx\,dy\,dt\,ds \geq 0.$$

Now change the variable

$$\bar{x} = \frac{x+y}{2}, \quad \bar{t} = \frac{t+s}{2}, \quad \bar{y} = \frac{x-y}{2}, \quad \bar{s} = \frac{t-s}{2}$$

then (4) becomes

(5). $\int_0^\infty \int_{-\infty}^{\infty} \Big[\int_0^\infty \int_{-\infty}^{\infty} |u(\bar{x}+\bar{y}, \bar{t}+\bar{s}) - \bar{u}(\bar{x}-\bar{y}, \bar{t}-\bar{s})| \, \varphi_t(\bar{x},\bar{t})$
$+ \operatorname{Sgn}(u(\bar{x}+\bar{y}, \bar{t}+\bar{s}) - \bar{u}(\bar{x}-\bar{y}, \bar{t}-\bar{s}))(f(u(\bar{x}+\bar{y}, \bar{t}+\bar{s})) - f(\bar{u}(\bar{x}-\bar{y}, \bar{t}-\bar{s})))\varphi_x(\bar{x},\bar{t}) \, d\bar{x}\, d\bar{t} \Big] \times$
$\eta_\varepsilon(\bar{y}) \cdot \eta_\varepsilon(\bar{s}) \, d\bar{y}\, d\bar{s} \geq 0.$

Now $(a,b) \to |a-b|$ and $(a,b) \to \operatorname{Sgn}(a-b)(f(a)-f(b))$ are Lipschitz continuous functions. If $\bar{y}, \bar{s} \to 0$, $u(\bar{x}+\bar{y}, \bar{t}+\bar{s}) \to u(\bar{x},\bar{t})$, $\bar{u}(\bar{x}-\bar{y}, \bar{t}-\bar{s}) \to \bar{u}(\bar{x},\bar{t})$ in L^1_{loc}. Since η_ε is Friedrich's mollifier, we get as $\varepsilon \to 0$

(6) $\int_0^\infty \int_{-\infty}^{\infty} |u(\bar{x},\bar{t}) - \bar{u}(\bar{x},\bar{t})| \, \varphi_t(\bar{x},\bar{t}) + \operatorname{Sgn}((u(\bar{x},\bar{t}) - \bar{u}(\bar{x},\bar{t}))(f(u(\bar{x},\bar{t})) - f(\bar{u}(\bar{x},\bar{t})))\varphi_x(\bar{x},\bar{t})$

Now let
$a(x,t) = |u(x,t) - \bar{u}(x,t)|$, $b(x,t) = \operatorname{Sgn}(u(x,t) - \bar{u}(x,t))(f(u(x,t)) - f(\bar{u}(x,t)))$
and write $\bar{x} = x$ and $\bar{t} = t$, we get (6) as

(6)' $\int_0^\infty \int_{-\infty}^{\infty} [a(x,t)\varphi_t(x,t) + b(x,t)\,\varphi_x(x,t)] \, dx\, dt \geq 0$

Step 3. to show L^1 contraction:

$\int_{-\infty}^{\infty} |u(x,t) - \bar{u}(x,t)| \, dx \leq \int_{-\infty}^{\infty} |u(x,s) - \bar{u}(x,s)| \, dx$ for a.e. $0 \leq s < t$.

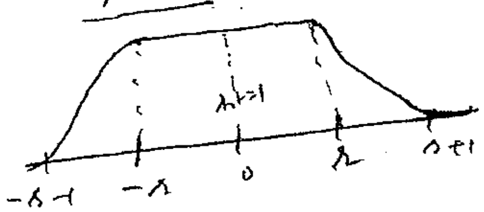
To do this we choose φ suitably. Let $0 < s < t$, $s > 0$

$\varphi(x,t) = \alpha(x) \cdot \beta(t)$

$\alpha : R \to R$, smooth

$\alpha(x) = \begin{cases} 1 & |x| < r \\ 0 & |x| \geq r+1 \end{cases}$

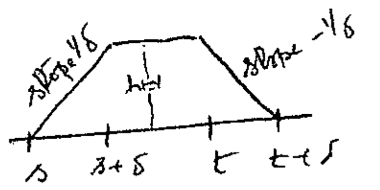
for: α:



$|\alpha'(x)| \leq 2$

$\beta : R \to R$, ~~smooth~~ Lipschitz

$\beta(\tau) = \begin{cases} 0 & 0 \leq \tau \leq s, \text{ or } \tau > t+\delta \\ 1 & s+\delta \leq \tau \leq t \\ \text{linear on } [s, s+\delta] \text{ and } [t, t+\delta] \end{cases}$



For: $0 < \delta < t-s$

$\frac{1}{\delta} \int_s^{s+\delta} \int_{-\infty}^{\infty} a(x,\tau)\alpha(x) \, dx\, d\tau - \frac{1}{\delta} \int_t^{t+\delta} \int_{-\infty}^{\infty} a(x,\tau)\alpha(x) \, dx\, d\tau$
$+ \int_s^{t+\delta} \int_{r \leq |x| \leq r+1} b(x,\tau)\alpha'(x)\beta(\tau) \, dx\, d\tau \geq 0$

let $s \to \infty$, we get

$$\frac{1}{\delta} \int_{t}^{t+\delta} \int_{-\infty}^{\infty} a(x,\tau)\, \alpha(x)\, dx\, d\tau \le \frac{1}{\delta} \int_{s}^{s+\delta} \int_{-\infty}^{\infty} a(x,\tau)\, \alpha(x)\, dx\, d\tau$$

Now $\delta \to 0$, we get

$$\int_{-\infty}^{\infty} a(x,t)\, dx \le \int_{-\infty}^{\infty} a(x,s)\, dx.$$

which is L^1 contraction.