Distribution theory.

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1. Introduction.

Let $I = (a,b)$ and f and g are differentiable function on I and continuous in $\bar{I}$. Then the integration by parts gives

$$\int_a^b f'(t)g(t)\,dt = -\int_a^b f(t)g'(t)\,dt + f(b)g(b) - f(a)g(a).$$

In particular, if $g(b)=g(a)=0$, Then

$$\int_a^b f'(t)g(t)\,dt = -\int_a^b f(t)g'(t)\,dt$$

Using this formula, we would like to extend the concept of derivative of a function which need not be differentiable.

In order to do this we need few definitions and properties of functions.

Let $\Omega \subset \mathbb{R}^n$ be an open set and $u : \Omega \to \mathbb{R}$ be a continuous function. Define the support of u by

$$\text{Supp}\, u = \overline{\{x \in \Omega : u(x) \neq 0\}}$$

We say that u has compact support in Ω if $\text{Supp}\, u \subset \Omega$ is compact.

Let $1 \leq k \leq \infty$, define

$$C^k(\Omega) = \{u : \Omega \to \mathbb{R}(\mathbb{C}) : u \text{ is } k\text{-times cont. diff function in } \Omega\}.$$

$C_0^k(\Omega) = \{u \in C^k(\Omega) : \text{supp}\, u \text{ is compact in } \Omega\}$.

For a compact set $K \subset \Omega$, $0 \leq m \leq k$, $u \in C^k(\Omega)$, define the seminorms

$$p_{K,m,\Omega}(u) = \sum_{|\alpha| \leq m} \sup_{x \in \Omega} |\partial^\alpha u(x)|$$

where $\alpha = (\alpha_1 \cdots \alpha_n) \in \mathbb{N}^n \times (0)^n$,

$|\alpha| = \alpha_1 + \cdots + \alpha_n$

$$\partial^\alpha u = \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \quad \alpha^{th} \text{ derivative of } u.$$

Clearly $p_{K,m,\Omega}(u) \geq 0$ and $p_{K,m,\Omega}(u) = 0$ iff $\partial^\alpha u(x) = 0 \ \forall x \in K, |\alpha| \leq m$.

$$p_{K,m,\Omega}(u_1 + u_2) \leq p_{K,m,\Omega}(u_1) + p_{K,m,\Omega}(u_2)$$

$$p_{K,m,\Omega}(bu) = |b|\, p_{K,m,\Omega}(u)$$

$\forall\ b \in \mathbb{R}(\mathbb{C})$.

Hence $\{p_{K,m,\Omega}\}$ forms a family of seminorms on $C_b^k(\Omega)$ when $0 \leq m \leq k$

$$C_b^k(\Omega) = \{u \in C^k(\Omega) ; p_{K,m,\Omega}(u) < \infty \ \forall K, 0 \leq m \leq k\}$$

This defines a metric topology on $C_b^k(\Omega)$ by "a sequence $\{u_\ell\}$ converges to u in this topology if $\forall\, \epsilon>0$, $\forall\, K\subset\Omega$ compact, $\forall\, m\in\mathbb{N}$,

$$p_{K,m,\Omega}(u_\ell-u)\to 0 \text{ as } \ell\to\infty.$$

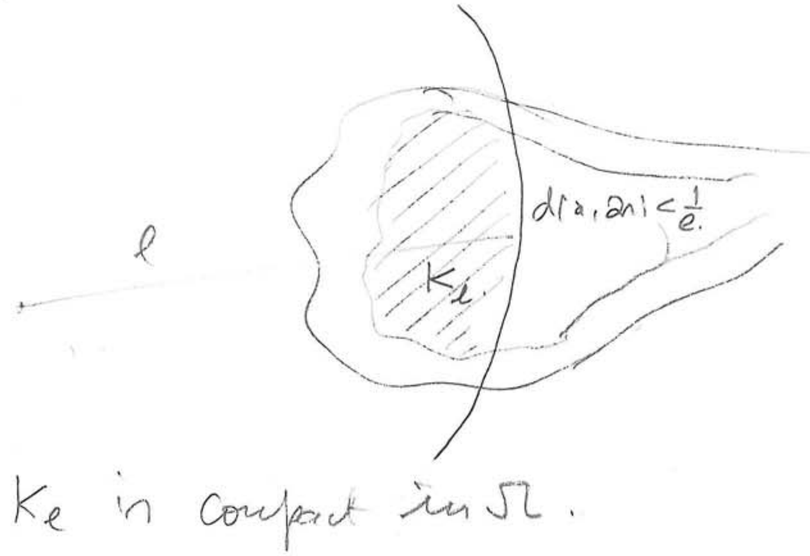
<u>Proposition</u>: $C^k_b(\Omega)$ is a complete metric space with respect to this topology.

<u>Proof</u>: For $x\in\mathbb{R}^n$, define

$$d(x,\partial\Omega)=\inf_{y\in\partial\Omega}|x-y|$$

$\forall\, \ell\in\mathbb{N}$, define

$$K_\ell=\{x\in\Omega:\ d(x,\partial\Omega)<\tfrac{1}{\ell},\ |x|\le \ell\}.$$



Then $\mathring{K}_\ell\subset\mathring{K}_{\ell+1}$, K_ℓ is compact in Ω.

$$\Omega=\bigcup_{\ell=1}^{\infty}K_\ell=\bigcup_{\ell=1}^{\infty}\mathring{K}_\ell$$

Hence if $K\subset\Omega$ is a compact set then $\exists$ $\{\mathring{K}_\ell\}$ is a cover for K and then $\exists$ a finite subcover $\{\mathring{K}_\ell\}_{\ell=1}^{L}$.

Since $\mathring{K}_\ell\subset\mathring{K}_{\ell+1}$, hence

$$K\subset\bigcup_{\ell=1}^{L}\mathring{K}_\ell\subset\mathring{K}_L\subset K_L.$$

Hence every compact set K in Ω is contained in some K_ℓ.
Hence

$$p_{K,m,\Omega}(u) \leq p_{K_\ell,m,\Omega}(u).$$

As a consequence of this if $u_s \to u$ in C^k_b iff $\forall \ell \geq 1$, $\forall m \leq k$,

$$p_{K_\ell,m,\Omega}(u_s - u) \to 0 \text{ as } s \to \infty.$$

Hence $C^k_b(\Omega)$ is defined by the countable family of seminorm $\{p_{K_\ell,m,\Omega}\}_{\ell,m}$;

Define

$$d(u,v) = \sum_{m=0}^{k} \sum_{\ell=1}^{\infty} \frac{1}{2^{\ell+m}} \frac{p_{K_\ell,m,\Omega}(u-v)}{1+p_{K_\ell,m,\Omega}(u-v)}.$$

Then d is a metric on C^k_b and the by the above analysis the metric topology generated by d coincides with the above topology on C^k_b.

<u>Claim</u>: (C^k_b, d) is a complete metric space.

Let $\{u_s\}$ be a Cauchy sequence. Then $\forall$ ℓ, m,

$$p_{K_\ell,m,\Omega}(u_s - u_{s'}) \to 0 \quad \text{as } s, s' \to \infty.$$

Hence on each K_ℓ, $u_s \to u^{(\ell)} \in C^k(\mathring{K}_\ell)$. Since $\mathring{K}_\ell \subset \mathring{K}_{\ell+1}$, $\Rightarrow u^{(\ell+1)}|_{\mathring{K}_\ell} = u^{(\ell)}$. Hence $u^{(\ell)} = u$ is independent of ℓ and $u_s \to u \in C^k_b(\Omega)$... this proves the proposition.

<u>Topology on $C_0^k(\Omega)$</u>. $C_0^k(\Omega) \subset C_b^k(\Omega)$ and hence it induces a topology on $C_0^k(\Omega)$ under which $C_0^k(\Omega)$ is not complete.

<u>Ex</u>: Let $k=0$, $\Omega = \mathbb{R}$,

$$u_n(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } |x| \leq n\pi \\ 0 & \text{if } |x| \geq n\pi \end{cases}$$

$$\sup_K |u_n(x) - u_m(x)| \leq \frac{1}{n} + \frac{1}{m} \to 0 \text{ as } n, m \to \infty.$$

$\forall\ K \subset \mathbb{R}$, compact, and $\lim_{n\to\infty} u_n(x) = u(x) = \frac{\sin x}{x}$.

$u_n \in C_b^0(\mathbb{R})$, $u \notin C_0^0(\mathbb{R})$

In order to define a topology on $C_0^k(\Omega)$ compatible with $C_b^k(\Omega)$ and preserving the compactness, we define it as follows:

<u>Definition</u>. <u>Topology on $C_0^k(\Omega)$</u>. Let $\{u_\ell\}$ be a sequence in $C_0^k(\Omega)$. We say that u_ℓ is a Cauchy sequence if

(i) $\exists$ a compact set $K \subset \Omega$ such that
$$\operatorname{supp} u_\ell \subset K \ \forall \ell.$$

(ii) $\{u_\ell\}$ is Cauchy in $C_b^k(\Omega)$.

As an immediate consequence of this we have the following

<u>Proposition</u>. W.r.t. the above topology, $C_0^k(\Omega)$ is sequentially complete.

Proof: Let $\{u_\ell\}$ be a Cauchy sequence in $C_0^k(\Omega)$. Then $u_\ell \to u$ in $C_b^k(\Omega)$. Since $\operatorname{Supp} u_\ell \subset K$ $\forall \ell$ $\Rightarrow \operatorname{Supp} u \subset K \Rightarrow u \in C_0^k(\Omega)$. This proves the Proposition

Mollifiers: Let $\rho \in C_0^\infty(\mathbb{R}^n)$ such that

(i) $\rho(x) \geq 0$

(ii) $\int_{\mathbb{R}^n} \rho(x)\,dx = 1$

[For example take $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi \geq 0$. define $\rho(x) = \frac{\varphi(x)}{\int \varphi(x)\,dx}$]

For $\epsilon > 0$, define $\rho_\epsilon(x) = \frac{1}{\epsilon^n} \rho(x/\epsilon)$ called a Mollifier.

Properties: (i) $\rho_\epsilon \geq 0$.

(ii) If $\operatorname{Supp} \rho \subset B(R)$, Then $\operatorname{Supp} \rho_\epsilon \subset B(\epsilon R)$

where $B(R) = \{x \in \mathbb{R}^n ; |x| < R\}$ is an open ball of radius R

(iii) $\int_{\mathbb{R}^n} \rho_\epsilon(x)\,dx = 1$

Proof. Since $\rho_\epsilon(x) = \frac{1}{\epsilon^n} \rho(x/\epsilon)$, hence if $|x| > R$, Then $\rho(x) = 0$ $\Rightarrow \left|\frac{x}{\epsilon}\right| > R \Rightarrow \rho_\epsilon(x) = 0$ iff $|x| > \epsilon R$, $\rho\left(\frac{x}{\epsilon}\right) = 0$

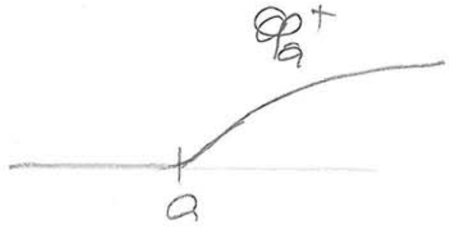
$\Rightarrow \operatorname{Supp} \rho_\epsilon \in B(\epsilon R)$.

$$\int_{\mathbb{R}^n} \rho_\epsilon(x)\,dx = \frac{1}{\epsilon^n} \int_{\mathbb{R}^n} \rho(x/\epsilon)\,dx = \int_{\mathbb{R}^n} \rho(x)\,dx = 1$$

Construction of functions in $C_0^\infty(\mathbb{R}^n)$.

Let $a \in \mathbb{R}$ and define

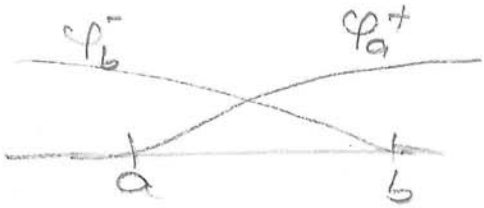
$$\varphi_a^+(t) = \begin{cases} e^{-1/(t-a)} & \text{if } t > a. \\ 0 & \text{if } t \le a \end{cases}$$



$$\varphi_a^-(t) = \begin{cases} e^{\frac{1}{t-a}} & \text{if } t < a \\ 0 & \text{if } t \geq a \end{cases}$$

Then $\varphi_a^{\pm} \in C^\infty(\mathbb{R})$ with $\varphi_a^{\pm} \geq 0$. Let $a < b$, define

$$\psi_{a,b}(t) = \varphi_a^+(t)\,\varphi_b^-(t)$$



Then $\psi_{a,b} \in C^\infty(\mathbb{R})$ and $\operatorname{Supp}\psi_{a,b} \subset [a,b]$ and $\psi_{a,b} > 0$ in (a,b)

Consequences.

(1) $C_0^\infty(B(R)) \neq \phi$.

For define $\varphi(x) = \psi_{-\frac{R^2}{4}, \frac{R^2}{4}}(|x|^2)$, then $\varphi(x) > 0$ for $|x| < R/2$ and $\varphi(x) = 0$ if $|x| \geq R/2$.

(2). $\exists\ \varphi \in C_0^\infty(\mathbb{R}^n)$ such that

$$0 \leq \varphi(x) \leq 1$$

$$\varphi(x) = \begin{cases} 1 & \text{if } |x| < a \\ 0 & \text{if } |x| > b \end{cases} \qquad (\text{here } 0 < a < b).$$

Define
$$h_{a,b}(t) = \frac{\int_t^\infty \psi_{a,b}(\sigma)\,d\sigma}{\int_{\mathbb{R}^n} \psi_{a,b}(\sigma)\,d\sigma}.$$

Clearly $0 \le h_{a,b} \le 1$, and $h_{a,b}(t) = 0$ if $t > b$ and $h_{a,b}(t) = 1$ if $t \le a$. Define

$$\varphi(x) = h_{a^2, b^2}(|x|^2).$$ Then φ is a required function.

Convolution. Let $f, g \in L^1(\mathbb{R}^n)$, define

$$(f*g)(x) = \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy$$

Properties.

(1) $f*g \in L^1$

$$\|f*g\|_{L^1} = \int_{\mathbb{R}^n} |f*g|\, dx \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x-y)|\, |g(y)|\, dy$$

$$\le \|f\|_1 \|g\|_1 .$$

(2) Young's inequality.

$f \in L^1$, $g \in L^q$, $f*g \in L^q$. $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$

$$|(f*g)(x)| \le \int_{\mathbb{R}^n} |f(x-y)|\, |g(y)|\, dy$$

$$= \int_{\mathbb{R}^n} |f(x-y)|^{1/p}\, |f(x-y)|^{1/q}\, |g(y)|$$

$$\le \left(\int |f(x-y)|\right)^{1/p} \left(\int |f(x-y)|\, |g(y)|^q\right)^{1/q}$$

$$\|f*g\|_{L^q}^q = \int |f*g|^q\, dx \le \|f\|_1^{q/p} \left(\int\int |f(x-y)|\, |g(y)|^q\right)$$

$$= \|f\|_1^{1+q/p} \|g\|_{L^q}^q .$$

$$\|f*g\|_{L^2} \le \|f\|_{L^1} \|g\|_{L^2}.$$

(3) Let A and B are closed sets such that $f(x) = 0$ for $x \notin A$, $g(x) = 0$ $x \notin B$. Then

$$(f*g)(x) = 0 \quad \forall\, x \notin A+B.$$

<u>Proof</u>: Let $x \notin A+B$, then $x-y \notin A$ $\forall$ $y \in B$. hence $f(x-y) = 0$ $\forall$ $y \in B$. Hence $f(x-y)g(y) = 0$ $\forall$ $y \in B$

$$(f*g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy$$

$$= \int_B f(x-y)\,g(y)dy = 0.$$

In particular, if A and B are compact, Then $f*g$ vanishes outside the compact set $A+B$

(4) If g is in $C^k(\mathbb{R}^n)$ such that $\partial^\alpha g \in L^\infty$ $\forall$ $|\alpha| \le k$, Then $f*g \in C^k(\mathbb{R}^n)$

$$(f*g)(x) = \int f(y)\,g(x-y)\,dy$$

$$J_h = \frac{(f*g)(x+he_i) - (f*g)(x)}{h} = \int f(y)\left(\frac{g(x+he_i-y) - g(x-y)}{h}\right)dy$$

$$\frac{g(x-y+he_i) - g(x-y)}{h} = \int_0^1 \frac{\partial g}{\partial x_i}(x+the_i)dt$$

$$\therefore I_h = \int_{\mathbb{R}^n} f(y) \int_0^1 \frac{\partial g}{\partial x_i}(x-y+the_i)\, dt\, dx$$

$$= \int_0^1 \left(\int_{\mathbb{R}^n} f(y) \frac{\partial g}{\partial x_i}(x-y+the_i)\, dx \right) dt.$$

Since $\left\|\frac{\partial g}{\partial x_i}\right\|_\infty \leq M$, hence by dominated convergence theorem, letting $h \to 0$ to obtain

$$\frac{\partial}{\partial x_i}(f*g) = \int_{\mathbb{R}^n} f(y) \frac{\partial g}{\partial x_i}(x-y)\, dy = f * \frac{\partial g}{\partial x_i}.$$

Similarly

$$\partial^\alpha (f*g) = (f * \partial^\alpha g) \quad \forall\ |\alpha| \leq k.$$

(5) $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ $\forall\ 1 \leq p < \infty$

Proof: Let $f \in L^p(\mathbb{R}^n)$ and $R > 0$. define

$$f_R = \begin{cases} f & \text{if } |x| < R \\ 0 & \text{if } |x| > R, \end{cases}$$

then

$$\|f - f_R\|_p^p = \int_{\mathbb{R}^n} |f(x) - f_R(x)|^p dx$$

$$= \int_{|x|>R} |f(x)|^p dx \to 0 \quad \text{as } R \to \infty.$$

Hence w.l.o.g. can assume that f vanishes outside a ball of radius R.

Let $N > 0$, define

$$f_N(x) = \begin{cases} f(x) & \text{if } |f| \leq N \\ 0 & \text{if } |f| > N. \end{cases}$$

Then

$$\int |f(x) - f_N(x)|^p \, dx = \int_{|f|>N} |f(x)|^p \, dx$$

By Chebyshev's inequality

$$N^p \, |\{x : |f| > N\}| \le \int_{|f|>N} |f|^p \le \int |f|^p = \|f\|_p^p .$$

$$\Rightarrow \quad |\{x : |f| > N\}| \le \left(\frac{\|f\|_p}{N}\right)^p \to 0 \text{ as } N \to \infty .$$

Let $E_N = \{x : |f| > N\}$, then $|E_N| = \text{meas}(E_N) \to 0$ as $N \to \infty$ and hence

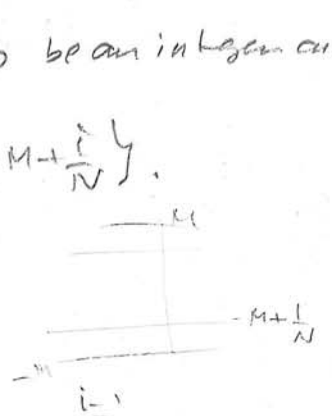
$$\lim_{N \to \infty} \int_{E_N} |f|^p \, dx = 0$$

Hence w.l.g (without loss of generality) we can assume that f is bounded and vanishes outside a ball R.

Let $|f| \le M$, M an integer. Let $N > 0$ be an integer and define

$$E_{i,N} = \{x \in B(R) : -M + \tfrac{i-1}{N} \le f(x) \le -M + \tfrac{i}{N}\} .$$

$$f_N(x) = \sum_{i=1}^{2MN} \left(-M + \frac{i-1}{N}\right) \chi_{E_{iN}}(x) .$$



$\frac{i}{N} - M = M.$

$i = 2MN.$

Then

$$|f(x) - f_N(x)| \le \frac{1}{N}$$

and hence

$$\int_{B(R)} |f - f_N|^p \, dx \le \frac{|B(R)|}{N^p} \to 0 \text{ as } N \to \infty .$$

As a consequence of this, we can assume that f is a characteristic function χ_E with $|E| < \infty$. For $\epsilon > 0$, choose a compact set K and an open set U s.t.

$$\begin{cases} K \subset E \subset U \\ |U \setminus K| < \epsilon . \end{cases}$$

Let $\eta_\epsilon(x) = \dfrac{d(x, U^c)}{d(x, K) + d(x, U^c)}$

Then $0 \le \eta_\epsilon \le 1$, $\eta_\epsilon(x) = 1$ if $x \in K$, $\eta_\epsilon(x) = 0$ if $x \notin U$. η_ϵ is a continuous function. Hence

$$\int |\chi_E - \eta_\epsilon|^p dx = \int_{U \setminus K} |\chi_E - \eta_\epsilon|^p dx \le |U \setminus K| < \epsilon$$

$\Rightarrow \eta_\epsilon \to \chi_E$ as $\epsilon \to 0$.

<u>Conclusion</u>. $C_0^0(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ is dense.

Let $\{\rho_\epsilon\}$ be a mollifying sequence and $f \in C_0^0(\mathbb{R}^n)$.

Then $\rho_\epsilon * f \in C_0^\infty(\mathbb{R}^n)$ and

$$\begin{aligned} |(\rho_\epsilon * f)(x) - f(x)| &= \left| \int_{\mathbb{R}^n} f(x-y)\rho_\epsilon(y) - f(x) \right| \\ &= \left| \int_{\mathbb{R}^n} f(x-y)\rho_\epsilon(y) - f(x)\int_{\mathbb{R}^n} \rho_\epsilon(y) \right| \\ &\le \int_{\mathbb{R}^n} |f(x-y) - f(x)| \rho_\epsilon(y)\, dy \\ &= \int_{\mathbb{R}^n} |f(x-\epsilon y) - f(x)| \rho(y)\, dy . \end{aligned}$$

Since f and ρ are bounded and have compact support, Therefore by dominated convergence then

$$\lim_{\epsilon \to 0} \sup |\rho_\epsilon * f - f| = 0.$$

Now chose $R > 0$ such that $\forall \epsilon$, $\operatorname{Supp} \rho_\epsilon * f \subset B(R)$, $\operatorname{Supp} f \subset B(R)$. Then

$$\int |\rho_\epsilon * f - f|^p \, dx \le |B(R)| \sup_x |(\rho_\epsilon * f)(x) - f(x)|$$
$$\to 0 \text{ as } \epsilon \to 0.$$

This proves the density result.

Corollary: Let $f \in L^1_{loc}(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} f \varphi = 0 \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n).$$

Then $f = 0$ a.e.

Proof: Let $\psi \in C_0^\infty(\mathbb{R}^n)$, Then $\psi f \in L^1$ and $\forall \varphi \in C_0^\infty(\mathbb{R}^n)$

$$\int (f\psi)\varphi = \int f(\psi\varphi) = 0.$$

Since $C_0^\infty(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ is dense, hence $f\psi = 0$ a.e.

Now choose $\psi = 1$ in $B(R)$ implies $f = 0$ a.e.

This proves the Corollary.

Definition: Let A be a set and $\epsilon > 0$. Define the ϵ-nhd A_ϵ of A by

$$A_\epsilon = \{x \in \mathbb{R}^n : d(x, A) < \epsilon\} = \bar{A} + B(\epsilon).$$
$$= \{x \in \mathbb{R}^n : d(x, \bar{A}) < \epsilon\}$$

Lemma: Let K be a compact subset and $\epsilon > 0$. Then $\exists\ \varphi \in C_0^\infty(\mathbb{R}^n)$ such that

$$\varphi = \begin{cases} 1 & \text{on } K \\ 0 & \text{on } \mathbb{R}^n \setminus K_{2\epsilon}. \end{cases}$$

Proof: Let $\{\rho_\epsilon\}$ be a mollifying sequence that supp $\rho \subset B(1)$. Let

$$\varphi = \rho_\epsilon * \chi_{K_\epsilon}.$$

Then

$$\operatorname{Supp} \varphi \subset \operatorname{Supp} \rho_\epsilon + \operatorname{Supp} \chi_{K_\epsilon} = B(\epsilon) + K + B(\epsilon) = K_{2\epsilon}$$

$$\varphi(x) = \int_{\mathbb{R}^n} \rho_\epsilon(y)\, \chi_{K_\epsilon}(x-y)\, dy = \int_{B(\epsilon)} \chi_{K_\epsilon}(x-y)\, \rho_\epsilon(y)\, dy$$

Hence if $x \in K$, $|y| \le \epsilon \Rightarrow x - y \in K_\epsilon \Rightarrow \chi_{K_\epsilon}(x-y) = 1$, hence

$$\varphi(x) = \int_{B(\epsilon)} \rho_\epsilon(y)\, dy = 1.$$

This proves the Lemma.

With these preliminaries, let us start the theory of distributions:

Motivation: Integration by parts.

Let $f \in C^1(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$. Then $f\varphi \in C_0^1(\Omega)$. Let $\chi \in C_0^\infty(\Omega)$ such that $\chi = 1$ on $\operatorname{Supp}(\varphi)$. This χ exists by the previous Lemma by taking ϵ sufficiently small. Hence $\chi\varphi = \varphi \Rightarrow f\varphi = (\chi f)\varphi$.

$$\frac{\partial}{\partial x_i}(\chi f)\varphi = (\chi\varphi)\frac{\partial f}{\partial x_i} + f\frac{\partial \chi}{\partial x_i}\varphi$$

$$= \varphi\frac{\partial f}{\partial x_i} + 0 \quad \text{since } \chi = 1 \text{ on support of } \varphi.$$

Also $f\psi \in C_0^1(\mathbb{R}^n)$. Hence

$$\int_\Omega \frac{\partial f}{\partial x_i}\varphi\,dx = \int_{\mathbb{R}^n} \frac{\partial}{\partial x_i}(f\psi)\,\varphi\,dx.$$

$$= -\int_{\mathbb{R}^n} f\psi \frac{\partial \varphi}{\partial x_i}$$

$$= -\int_\Omega f \frac{\partial \varphi}{\partial x_i} \qquad \text{since } \psi \frac{\partial \varphi}{\partial x_i} = \frac{\partial \varphi}{\partial x_i}$$

$\therefore$ we have

$$\int_\Omega \frac{\partial f}{\partial x_i}\varphi\,dx = -\int_\Omega f \frac{\partial \varphi}{\partial x_i}$$

If $f \in C^k(\Omega)$, $|\alpha| \le k$, then

$$\int_\Omega (\partial^\alpha f)\,\varphi = (-1)^{|\alpha|} \int_\Omega f\, \partial^\alpha \varphi.$$

For $f \in L^1_{loc}(\Omega)$, define the linear operator L_f on $C_0^\infty(\Omega)$ by

$$L_f : C_0^\infty(\Omega) \longrightarrow \mathbb{R}\ (\mathbb{C})$$

by

$$L_f(\varphi) = \int_\Omega f\varphi\,dx$$

<u>Observation</u>

(1) $L_f(a\varphi_1 + b\varphi_2) = a L_f(\varphi_1) + b L_f(\varphi_2)$ – linear.

(2) Let $K \subset \Omega$ be compact and denote

$$C_0^\infty(K) = \{\varphi \in C_0^\infty(\Omega) : \operatorname{Supp}\varphi \subset K\}$$

Then $\forall\ \varphi \in C_0^\infty(K)$, we have

$$|L_f(\varphi)| \le \int_K |f||\varphi| \le p_{K,0,\Omega}(\varphi)\, \|f\|_{L^1(K)}.$$

Hence if $\varphi_\ell \in C_0^\infty(\Omega)$ s.t. $\varphi_\ell \to \varphi$ in $C_0^\infty(\Omega)$,

then $|L_f(\varphi_\ell) - L_f(\varphi)| \le p_{K,0,\Omega}(\varphi_\ell - \varphi)\, \|f\|_{L^1(K)}$

$\to 0$ as $\ell \to \infty$.

Hence L_f is a continuous linear map on $C_0^\infty(\Omega)$.

(3) Let $f_1, f_2 \in L^1_{loc}(\Omega)$ be such that $L_{f_1} = L_{f_2}$.

Then $\forall\ \varphi \in C_0^\infty(\Omega)$

$$L_{f_1}(\varphi) = L_{f_2}(\varphi) \Rightarrow \int_\Omega (f_1 - f_2)\varphi = 0 \quad \forall\ \varphi \in C_0^\infty(\Omega)$$

$\Rightarrow f_1 = f_2$ a.e.

Hence $L_{f_1} = L_{f_2} \iff f_1 = f_2$ in $L^1_{loc}(\Omega)$.

Hence each $f \in L^1_{loc}(\Omega)$ can be looked upon as a continuous linear map on $C_0^\infty(\Omega)$.

Now assume that $f \in C^k(\Omega)$, then $\partial^\alpha f \in C^0(\Omega)$

$\Rightarrow \partial^\alpha f \in L^1_{loc}(\Omega)$. Hence $\forall\ \varphi \in C_0^\infty(\Omega)$

$$L_{\partial^\alpha f}(\varphi) = \int_\Omega \partial^\alpha f\, \varphi = (-1)^{|\alpha|} \int_\Omega f\, \partial^\alpha \varphi$$

$$= (-1)^{|\alpha|} L_f(\partial^\alpha \varphi).$$

i.e.

$$\begin{cases} L_{\partial^\alpha f}(\varphi) = (-1)^{|\alpha|} L_f(\partial^\alpha \varphi). \\ |L_{\partial^\alpha f}(\varphi)| = |L_f(\partial^\alpha \varphi)| \\ \qquad \le p_{K,m,\Omega}(\varphi) \, \|f\|_{L^1(K)} \end{cases}$$

$\forall \ \varphi \in C_0^\infty(K)$

With this observation, we generalize the concept of a function and derivatives as follows.

<u>Definition</u>: Let $u : C_0^\infty(\Omega) \to \mathbb{R}$ ($\mathbb{C}$) be a linear map. u is called a distribution if u is continuous w.r.t. $C_0^\infty(\Omega)$-topology. i.e. $\forall$ $K \subset\subset \Omega$, $\exists$ an integer m and a constant $C(K,m)$ such that $\forall$ $\varphi \in C_0^\infty(K)$

$$|\langle u, \varphi\rangle| \le C(K,m) \, p_{K,m,\Omega}(\varphi).$$

Denote $\mathcal{D}'(\Omega)$ to be all such linear maps and call it the set of "Distributions" in Ω.

Clearly $\mathcal{D}'(\Omega)$ is a vector space.

<u>Calculus on $\mathcal{D}'(\Omega)$.</u>

(1) <u>Derivatives on $\mathcal{D}'(\Omega)$</u>:- Let $u \in \mathcal{D}'(\Omega)$, $\alpha = (\alpha_1 \cdots \alpha_n)$.

Define $\partial^\alpha u : C_0^\infty(\Omega) \to \mathbb{R}$ ($\mathbb{C}$) by

$$\langle \partial^\alpha u, \varphi\rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha \varphi\rangle.$$

Claim: $\partial^\alpha u \in \mathcal{D}'(\Omega)$.

Proof: Let $K \subset \Omega$ be a compact set and $C(K,m)>0$ such that $\forall \varphi \in C_0^\infty(K)$

$$|\langle u, \varphi\rangle| \le C\, p_{K,m,\Omega}(\varphi).$$

Then $\forall \varphi \in C_0^\infty(K)$,

$$|\langle \partial^\alpha u, \varphi\rangle| = |(-1)^{|\alpha|}\langle u, \partial^\alpha\varphi\rangle|$$

$$= |\langle u, \partial^\alpha \varphi\rangle|$$

$$\le C\, p_{K,m,\Omega}(\partial^\alpha\varphi)$$

$$\le C\, p_{K,m+|\alpha|,\Omega}(\varphi)$$

Hence $\partial^\alpha u \in \mathcal{D}'(\Omega)$.

(2) Let $\psi \in C_0^\infty(\Omega)$, $u \in \mathcal{D}'(\Omega)$. define the linear map $\psi u : C_0^\infty(\Omega) \to \mathbb{R}\ (\mathbb{C})$ by

$$\langle \psi u, \varphi\rangle = \langle u, \psi\varphi\rangle.$$

Claim: $\psi u \in \mathcal{D}'(\Omega)$

Let $K \subset \Omega$ be compact and $C>0, m \ge 0$ be s.t. $\forall \varphi \in C_0^\infty(K)$

$$|\langle u, \varphi\rangle| \le C\, p_{K,m,\Omega}(\varphi).$$

Hence

$$|\langle \psi u, \varphi\rangle| = |\langle u, \psi\varphi\rangle| \le C\, p_{K,m,\Omega}(\psi\varphi)$$

since $\psi\varphi \in C_0^\infty(K)$.

From Leibnitz rule, $\exists$ $c_{\alpha,\beta} \in \mathbb{R}$ s.t.

$$\partial^\alpha(\varphi\psi) = \sum_{0\le\beta\le\alpha} c_{\alpha\beta}\, \partial^{\alpha-\beta}\psi\, \partial^\beta\varphi$$

$$\|\partial^\alpha(\varphi\psi)\|_\infty \le c \sum_{\beta\le\alpha} \|\partial^{\alpha-\beta}\psi\|_\infty \|\partial^\beta\varphi\|$$

$$\le c\, p_{\Omega,m,n}(\psi)\; p_{K,m,n}(\varphi)$$

$$\therefore \; |\langle \psi u, \varphi\rangle| \le \left(c\, p_{\Omega,m,n}(\psi)\right) p_{K,m,n}(\varphi)$$

Hence $\psi u \in \mathcal{D}'(\Omega)$.

(3) Convolution.

Motivation. Let $f, f_1, f_2 \in L^1(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ define

$$(\tau_x f)(y) = f(x-y)$$
$$\check{f}(y) = f(-y)$$

Then

$$(\tau_x(\tau_x f))(y) = (\tau_x f)(x-y)$$
$$= f(x+(x-y)) = f(y)$$
$$(\check{\check{f}})(y) = \check{f}(-y) = f(y)$$

Hence $\tau_x\tau_x = Id$, $V^2 = Id$.

$$(f_1 * f_2)(x) = \int f_1(y) f_2(x-y)\,dy = \int f_1(y)\, \tau_x f_2(y)\,dy$$
$$= \langle L_{f_1}, \tau_x f_2\rangle.$$

Now observe that if $\varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R}^n)$, then $\varphi_1 * \varphi_2 \in C_0^\infty(\mathbb{R}^n)$, $\check{\varphi}_1 \in C_0^\infty(\mathbb{R}^n)$. Using these informations we can define the convolution of $u \in \mathcal{D}'(\mathbb{R}^n)$ and a $\psi \in C_0^\infty(\mathbb{R}^n)$ as follows:

$$(u * \psi)(x) = \langle u, \tau_x \check{\psi} \rangle.$$

Claim:

(i) $u * \psi \in C^\infty(\mathbb{R}^n)$

(ii) $\partial^\alpha (u * \psi) = u * \partial^\alpha \psi = \partial^\alpha u * \psi$

(iii) $\forall \varphi \in C_0^\infty(\mathbb{R})$,
$\langle u * \rho_\epsilon, \varphi \rangle \to \langle u, \varphi \rangle$ as $\epsilon \to 0$.

Proof: Let $K = \operatorname{Supp} \psi$. Then $\operatorname{Supp} \partial^\alpha \psi \subset K$ $\forall \alpha$.
Let $x_0 \in \mathbb{R}^n$ and $R > 0$. Then $\forall x \in \overline{B(x_0, R)}$,

$$\operatorname{Supp} \tau_x \check{\psi} \subset -K - \overline{B(x_0, R)} = K_R$$

$$\partial_y^\alpha \tau_x \check{\psi}(y) = \partial^\alpha(\psi(x-y)) = (-1)^{|\alpha|} (\partial^\alpha \psi)(x-y)$$
$$= (-1)^{|\alpha|} (\tau_x \partial^\alpha \check{\psi})(y).$$

From the mean value theorem, $\forall x_1, x_2 \in \mathbb{R}^n$

$$|\partial^\alpha (\tau_{x_1} \check{\psi})(y) - \partial^\alpha (\tau_{x_2} \check{\psi})(y)| = |\partial^\alpha \psi(x_1 - y) - \partial^\alpha \psi(x_2 - y)|$$

$$= \left| \int_0^1 \frac{d}{dt} \partial^\alpha \psi(t x_1 + (1-t) x_2 - y)\, dt \right|$$

$$= \left| \int_0^1 \langle \nabla \partial^\alpha \psi(t x_1 + (1-t) x_2 - y), x_1 - x_2 \rangle\, dt \right|$$

$$\leq |x_1 - x_2|\, \| \partial^\alpha \nabla \psi \|_\infty.$$

Let $x_k \to x$, Then

$$\partial^\alpha \tau_{x_k}\psi \to \partial^\alpha \tau_x \psi \quad \text{uniformly}$$

Hence $\tau_{x_k}\psi \to \tau_x\psi$ in $C_0^\infty(\mathbb{R}^n)$, since Supp $\{\tau_{x_k}\psi, \tau_x\psi\}$ are all contained in fixed compact set.

Hence $(u*\psi)(x_k) = \langle u, \tau_{x_k}\psi\rangle \to \langle u, \tau_x\psi\rangle = (u*\psi)(x)$

Therefore $u*\psi \in C^0(\mathbb{R}^n)$.

$$\begin{aligned}(\partial^\alpha u * \psi)(x) &= \langle \partial^\alpha u, \tau_x\psi\rangle \\ &= (-1)^{|\alpha|}\langle u, \partial_y^\alpha(\tau_x\psi)\rangle \\ &= (-1)^{2|\alpha|}\langle u, \tau_x(\partial^\alpha\psi)\rangle \\ &= \langle u, \tau_x(\partial^\alpha u)\rangle = u * \partial^\alpha\psi(x).\end{aligned}$$

Let

$$\begin{aligned}\left[\frac{\tau_{x+he_i} - \tau_x}{h}\psi\right](y) &= \frac{\psi(x+he_i-y) - \psi(x-y)}{h} \\ &= \frac{1}{h}\int_0^1 \frac{d}{dt}\psi(x-y+the_i)\,dt \\ &= \int_0^1 \frac{\partial\psi}{\partial x_i}(x-y+the_i)\,dt \\ &\to \int_0^1 \frac{\partial\psi}{\partial x_i}(x-y) \quad \text{uniformly as } |h|\to 0. \\ &= \tau_x\left(\frac{\partial\psi}{\partial x_i}\right)(y)\end{aligned}$$

Since $\partial^\alpha(\tau_x\psi) = (-1)^{|\alpha|}\tau_x\partial^\alpha\psi$, hence

$$\left(\frac{\tau_{x+he_i} - \tau_x}{h}\right)\psi \to \tau_x\left(\frac{\partial\psi}{\partial x_i}\right)(y) \quad \text{in } C_0^\infty(\mathbb{R}^n).$$

Hence as $|h| \to 0$

$$\frac{(u*\psi)(x+he_i) - (u*\psi)(x)}{h} = \left\langle u, \frac{\tau_{x+he_i} - \tau_x}{h}\psi \right\rangle$$

$$\longrightarrow \left\langle u, \tau_x \frac{\partial \psi}{\partial x_i} \right\rangle$$

$$\therefore \quad \frac{\partial}{\partial x_i}(u*\psi) = \left\langle u, \tau_x \frac{\partial \psi}{\partial x_i} \right\rangle = u * \frac{\partial \psi}{\partial x_i}$$

Hence $u*\psi \in C^\infty(\mathbb{R}^n)$, and

$$\partial^\alpha (u*\psi) = \partial^\alpha u * \psi = u * \partial^\alpha \psi .$$

<u>Action of $u*\psi$</u>: Using the Rieman Sum, we have

Since $u*\psi \in C^\infty(\mathbb{R}^n)$, hence

$$L_{u*\psi}(\varphi) = \int_{\mathbb{R}^n} (u*\psi)(x)\varphi(x)\,dx$$

$$= \int_{\mathbb{R}^n} \langle u, \tau_x \psi \rangle \varphi(x)\,dx .$$

$$= \lim_{h \to 0} \left[h^n \sum_{i=1}^{N} \langle u, \tau_{x_i} \psi \rangle \varphi(x_i) \right]$$

when $N \cdot h =$ Constant

$$= \lim_{h \to 0} \left\langle u, h^n \left(\sum_{i=1}^{N} \tau_{x_i} \psi \right) \varphi(x_i) \right\rangle$$

$$I_N(y) = h^n \sum_{i=1}^{N} (\tau_{x_i} \psi)(y) \varphi(x_i) \in C_0^\infty(\mathbb{R}^n).$$

<u>Claim</u>: $I_N(y) \to \int_{\mathbb{R}^n} (\tau_x \psi)(y) \varphi(x)\,dx$ as $|h| \to 0$

in $C_0^\infty(\mathbb{R}^n)$.

$\mathrm{Supp}(\tau_{x_i}\psi) \subset -\mathrm{Supp}\,\psi + x_i \subset -K+Q$,

Where Q is a cube such that Support of $\varphi \subset Q$.

Hence

(i) $\mathrm{Supp}\, I_N \subset -K+Q$

(ii) $\partial^\alpha I_N(y) = (-1)^{|\alpha|} h^n \sum \tau_{x_i} \partial^\alpha \psi(y)$

(iii) $I_N(y) \longrightarrow \int_{\mathbb{R}^N} (\tau_x \psi)(y) \varphi(x)\, dx$ unif in y.

Hence

$$\langle L_{u*\psi}, \varphi\rangle = \langle u, \int_{\mathbb{R}^n} (\tau_x\psi)(y)\varphi(x)\,dx\rangle$$
$$= \langle u, \int_{\mathbb{R}^n} \psi(x-y)\varphi(x)\,dx\rangle$$
$$= \langle u, \check{\psi} * \varphi\rangle$$

$$\int_{\mathbb{R}^n} (u*\psi)\varphi = \langle u, \check{\psi}*\varphi\rangle = \langle u, \psi$$

As an immediate consequence of this we have the following: Let $\{\rho_\epsilon\}$ be a mollifying. Then $\{\check{\rho}_\epsilon\}$ is also a mollifying sequence. Hence $\check{\rho}_\epsilon * \varphi \to \varphi$ in $C_0^\infty(\mathbb{R}^n)$.

Hence

$$\langle u*\rho_\epsilon, \varphi\rangle = \langle u, \check{\rho}_\epsilon * \varphi\rangle$$
$$\to \langle u, \varphi\rangle \text{ as } \epsilon \to 0.$$

Hence $u*\rho_\epsilon \longrightarrow u$ weakly.

In conclusion we have the following

<u>Theorem</u>: Let $\varphi, \psi \in C_0^\infty(\Omega)$, $u \in \mathcal{D}'(\Omega)$. Then

(1) $\varphi u \in \mathcal{D}'(\Omega)$, $\partial^\alpha u \in \mathcal{D}'(\Omega)$ $\forall \alpha$.

(2) If $\Omega = \mathbb{R}^n$, $u * \psi \in C^\infty(\mathbb{R}^n)$ and is given by

$$(u*\psi)(x) = \langle u, \tau_x \check{\psi}\rangle.$$

$$\int (u*\psi)(x)\,\varphi(x)\,dx = \langle u, \check{\psi} * \varphi\rangle.$$

(3) $u * \rho_\epsilon \longrightarrow u$ weakly in $\mathcal{D}'(\mathbb{R}^n)$

(4) $\partial^\alpha (u*\varphi) = \partial^\alpha u * \varphi = u * \partial^\alpha \varphi$.

<u>Examples</u>:

1. $L^1_{loc}(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$ by $f \longmapsto L_f$.

2. Let $\mathcal{M}(\Omega) = \{$all finite Borel measures on $\Omega\}$

Then $\mathcal{M}(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$ by

$$\langle \mu, \varphi\rangle = \int_\Omega \varphi(x)\,d\mu(x)$$

$$|\langle \mu, \varphi\rangle| \le |\mu|(\Omega)\,\|\varphi\|_\infty .$$

3. <u>Dirac Mass</u>: Let $x_0 \in \Omega$. Then $\delta_{x_0} \in \mathcal{M}(\Omega)$ defined by

$$\int \varphi(x)\,d\delta_{x_0} = \varphi(x_0)$$

Then $\delta_{x_0} \in \mathcal{D}'(\Omega)$.

4.

(4) Let $f \in C^1(\Omega)$, then $L_f \in \mathcal{D}'(\Omega)$ and

$$\langle \frac{\partial}{\partial x_i} L_f, \varphi \rangle = -\langle L_f, \frac{\partial \varphi}{\partial x_i} \rangle$$
$$= -\int_\Omega f \frac{\partial \varphi}{\partial x_i}$$
$$= \int_\Omega \frac{\partial f}{\partial x_i} \varphi = \langle L_{\frac{\partial f}{\partial x_i}}, \varphi \rangle.$$

Hence $\frac{\partial}{\partial x_i} L_f = L_{\frac{\partial f}{\partial x_i}}$.

(5) If f is differentiable a.e. Then in general

$$\frac{\partial}{\partial x_i} L_f \neq L_{\frac{\partial f}{\partial x_i}}$$

For example, let f be the Cantor function on $[0,1]$. Then f is diff. a.e. and $\frac{\partial f}{\partial x} = 0$ a.e.

Hence $L_{\frac{\partial f}{\partial x}} = 0$ but $\frac{\partial}{\partial x} L_f \neq 0$.

(6) Let f be absolutely continuous function on $[a,b]$. i.e. $\exists\, g \in L^1(a,b)$ such that

$$f(x) = f(a) + \int_a^x g(t)\,dt$$

$$\langle \frac{d}{dx} L_f, \varphi \rangle = \langle L_f, -\frac{d\varphi}{dx} \rangle$$
$$= -f(a)\int_a^b \frac{d\varphi}{dx} - \int_a^b \left(\int_a^x g(t)\,dt\right) \frac{d\varphi}{dx}\,dx$$
$$= -\int_a^b \int_a^b \chi_{[a,x]}(t)\, g(t) \frac{d\varphi}{dx}\,dt\,dx$$

$$= -\int_a^b g(t) \left(\int_a^b \chi_{[a,x]}(t) \frac{d\varphi}{dx} dx \right) dt$$

$$= -\int_a^b g(t) \left(\int_t^b \frac{d\varphi}{dx} dx \right) dt$$

$$= \int_a^b g(t) \varphi(t) dt.$$

$$= \langle L_g, \varphi \rangle.$$

Hence $\frac{d}{dx} L_f = L_g = L_{\frac{df}{dx}}$.

(7) Let $f(x) = |x|$.

$$\langle \frac{d}{dx} L_f, \varphi \rangle = -\int_{\mathbb{R}} f \frac{d\varphi}{dx} = +\int_{-\infty}^{0} x \frac{d\varphi}{dx} - \int_0^{\infty} x \frac{d\varphi}{dx}$$

$$= -\int_{-\infty}^{0} \varphi(x) dx + \int_0^{\infty} \varphi(x) dx$$

$$= \int_{\mathbb{R}} A(x) \varphi(x) dx$$

where $A(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0. \end{cases}$

Hence $\frac{d}{dx} L_{|x|} = A(x) \in L^1_{loc}$.

(8) Let $H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$ – Heaviside function.

Then

$$\left\langle \frac{d}{dx} H, \varphi \right\rangle = -\left\langle H, \frac{d\varphi}{dx} \right\rangle = -\int_0^\infty \frac{d\varphi}{dx}\, dx = \varphi(0)$$

$$= \langle \delta_0, \varphi \rangle.$$

$$\frac{dH}{dx} = \delta_0.$$

$$\left\langle \frac{d^2H}{dx^2}, \varphi \right\rangle = \int_0^\infty H \frac{d^2\varphi}{dx^2} = \int_0^\infty \frac{d^2\varphi}{dx^2}\, dx = -\frac{d\varphi}{dx}(0)$$

$$= -\left\langle \delta_0, \frac{d\varphi}{dx} \right\rangle = \left\langle \frac{d\delta_0}{dx}, \varphi \right\rangle.$$

$$\frac{d^2H}{dx^2} = \frac{d\delta_0}{dx}.$$

(9) <u>Hilbert transform</u>:

$\frac{1}{x} \notin L^1_{loc}(\mathbb{R})$, hence $L_{1/x}$ does not make sense.

On the other hand define p.v. $\frac{1}{x} = \{\frac{1}{x}\}$ by.

$$\left\langle \left\{\frac{1}{x}\right\}, \varphi \right\rangle = \text{principal value} \int_{\mathbb{R}} \frac{\varphi(x)}{x}\, dx$$

$$= \lim_{\epsilon \to 0} \int_{|x| > \epsilon} \frac{\varphi(x)}{x}\, dx.$$

then $\left\{\frac{1}{x}\right\} \in \mathcal{D}'(\mathbb{R})$.

$$\int_{-\infty}^{-\epsilon} \frac{\varphi(x)}{x} + \int_{\epsilon}^{\infty} \frac{\varphi(x)}{x} = \int_{\epsilon}^{\infty} \frac{\varphi(x)-\varphi(-x)}{x}\,dx$$

$$= \int_{\epsilon}^{1} \frac{\varphi(x)-\varphi(-x)}{x}\,dx + \int_{1}^{\infty} \frac{\varphi(x)-\varphi(-x)}{x}\,dx.$$

$\varphi(x) = \varphi(0) + \varphi'(0)x + x^2\psi(x)$, $\psi \in C^\infty$

$\varphi(-x) = \varphi(0) - \varphi'(0)x + x^2\psi(-x)$

$\varphi(x) - \varphi(-x) = 2\varphi'(0)x + x^2(\psi(x) - \psi(-x))$

$$\int_{\epsilon}^{1} \frac{\varphi(x)-\varphi(-x)}{x} = 2\varphi'(0)(1-\epsilon) + \int_{\epsilon}^{1} x(\psi(x)-\psi(-x))\,dx.$$

Hence

$$\lim_{\epsilon\to 0} \int_{|x|>\epsilon} \frac{\varphi(x)}{x} = 2\varphi'(0) + \int_{0}^{1} x(\psi(x)-\psi(-x))\,dx$$
$$+ \int_{1}^{\infty} \frac{\varphi(x)-\varphi(-x)}{x}\,dx.$$

$\Longrightarrow$ p.v. $\frac{1}{x} \in \mathcal{D}'(\mathbb{R})$

(10) Let $0 \le \alpha < n$, and $f(x) = \frac{1}{|x|^\alpha}$. Then $f \in L^1_{loc}(\mathbb{R}^n)$ and hence $\frac{1}{|x|^\alpha} \in \mathcal{D}'(\mathbb{R}^n)$.

<u>Definition</u>: $u \in \mathcal{D}'(\Omega)$. u is said to be a function if $\exists$ $f \in L^1_{loc}(\Omega)$ such that $u = L_f$.

Leibnitz formula:

Let $P(\xi) = \sum_{|\alpha| \le m} a_\alpha \xi^\alpha$ be a polynomial of degree m.

For β a multi index, define

$$P^{(\beta)}(\xi) = \partial^\beta P(\xi)$$

Associate to P define the differential operator

$$P(\partial) = \sum a_\alpha \partial^\alpha$$

i.e. $$P(\partial) f = \sum_{|\alpha| \le m} a_\alpha \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

Let $f, g \in C^m(\Omega)$, Then

$$P(\partial)(fg) = \sum_{|\beta| \le m} \frac{(P^{(\beta)}(\partial) f)\, \partial^\beta g}{\beta!}$$

Proof:

$$\frac{\partial}{\partial x_i}(fg) = \frac{\partial f}{\partial x_i} g + f \frac{\partial g}{\partial x_i}$$

and hence by induction,

$$P(\partial)(fg) = \sum_{|\beta| \le m} (Q_\beta(\partial) f)\, \partial^\beta g$$

where $Q_\beta(\xi)$ are polynomials. In order to determine Q_β, take $f = e^{\langle x, \xi \rangle}$, $g = e^{\langle x, \eta \rangle}$, $\xi, \eta \in \mathbb{R}^n$. Then

$$\partial^\alpha f = \xi^\alpha e^{\langle x, \xi \rangle},$$

$$Q_\beta(\partial) f = Q_\beta(\xi) e^{\langle x, \xi \rangle}, \quad \partial^\beta g = \eta^\beta e^{\langle x, \eta \rangle}.$$

$$P(\partial)(fg) = P(\partial) e^{\langle x, \xi + \eta \rangle}$$

$$= P(\xi + \eta) e^{\langle x, \xi + \eta \rangle}.$$

Hence
$$e^{\langle x,\xi+\eta\rangle} P(\xi+\eta) = \left(\sum Q_\beta(\xi)\eta^\beta\right) e^{\langle x,\xi+\eta\rangle}$$

or
$$P(\xi+\eta) = \sum Q_\beta(\xi)\eta^\beta .$$

Hence
$$Q_\beta(\xi) = \frac{\partial^\beta P(\xi)}{\beta!}.$$

This proves the formula.

<u>Corollary</u>. Let $u \in \mathcal{D}'(\Omega)$, $\varphi \in C_0^\infty(\Omega)$, Then
$$P(\partial)(\varphi u) = \sum_{|\beta|\le m} \left(\frac{P^{(\beta)}(\partial)\varphi}{\beta!}\right) \partial^\beta u .$$

<u>Support of a distribution.</u>

Let $u \in \mathcal{D}'(\Omega)$ and $\Omega_1 \subset \Omega$ be an open set. We say that $u = 0$ on Ω_1 if $\forall \varphi \in C_0^\infty(\Omega_1)$,
$$\langle u, \varphi\rangle = 0$$
and denote $u|_{\Omega_1} = 0$.

If $W \subset \Omega_1$ is an open set. Then $u|_W = 0$ for $C_0^\infty(W) \subset C_0^\infty(\Omega_1)$. Also if $u|_{\Omega_1} = 0$ $u|_{\Omega_2} = 0$, then $u|_{\Omega_1 \cup \Omega_2} = 0$.

For $\varphi \in C_0^\infty(\Omega_1 \cup \Omega_2)$, $K = \operatorname{Supp}\varphi$. Then $K \subset \Omega_1 \cup \Omega_2$. Let $\{\varphi_i\}_{i=1}^2$ be a C_0^∞ partition of unity w.r.t. K and Ω_1 and Ω_2. i.e
$$\begin{cases} \varphi_1(x) + \varphi_2(x) = 1 & \forall x \in K. \\ \operatorname{Supp} \varphi_i \subset \Omega_i \end{cases}$$

Hence $\varphi(x) = \varphi_1(x)\varphi(x) + \varphi_2(x)\varphi(x)$

and $\varphi_i \varphi \in C_0^\infty(\Omega_i)$.

$$\langle u, \varphi \rangle = \langle u, \varphi_1 \varphi + \varphi_2 \varphi \rangle$$
$$= \langle u, \varphi_1 \varphi \rangle + \langle u, \varphi_2 \varphi \rangle$$
$$= 0$$

In view of this define

$$V = \{x \in \Omega : \exists \text{ open nbd } V_x \text{ of } x \text{ in } \Omega \text{ such that } u|_{V_x} = 0\}.$$

Then V is the largest open set on which $u|_V = 0$;

Define

$$\operatorname{Supp} u = \Omega \setminus V.$$

<u>Definition</u>: $u \in \mathcal{D}'(\Omega)$ is said to have compact support if $\operatorname{Supp} u$ is compact in Ω.

Denote $\mathcal{E}'(\Omega) = \{u \in \mathcal{D}'(\Omega) ; u \text{ has compact support}\}$.

<u>Properties of $\mathcal{E}'(\Omega)$</u>: Let $u \in \mathcal{E}'(\Omega)$, $K = \operatorname{Supp} u$.

(1) $\exists$ $C > 0$, $m \geq 0$ and a compact set K_1 with $K \subset \mathring{K}_1$ such that $\forall \varphi \in C_0^\infty(\Omega)$

$$|\langle u, \varphi \rangle| \leq C \, p_{K_1, m, \Omega}(\varphi) \quad \text{---} (*)$$

Conversely if $(*)$ holds, then $u \in \mathcal{E}'(\Omega)$.

<u>Proof</u>: Let $\epsilon > 0$ such that $K \subset K_{2\epsilon} \subset \overline{K_{2\epsilon}} \subset \Omega$.

Let $\psi \in C_0^\infty(K_{2\epsilon})$ such that $\psi = 1$ on K_ϵ.

Then $(1-\psi)|_{K_\epsilon} = 0$ and $u|_{{}^cK_\epsilon} = 0$. Let $\varphi \in C_0^\infty(\Omega)$, then

$$\varphi = \psi\varphi + (1-\psi)\varphi.$$

$$\langle u, \varphi\rangle = \langle u, \psi\varphi\rangle + \langle u, (1-\psi)\varphi\rangle$$

$$= \langle u, \psi\varphi\rangle$$

Since $\mathrm{Supp}\,(1-\psi)\varphi \subset {}^cK_\epsilon$, Since $\mathrm{Supp}\,\psi\varphi \subset \overline{K_{2\epsilon}} = K_1$ hence $\exists$ $c > 0, m \geq 0$ such that

$$|\langle u, \varphi\rangle| \leq |\langle u, \psi\varphi\rangle|$$

$$\leq c\, p_{K_1, m, n}(\psi\varphi)$$

$$\leq c_1\, p_{K_1, m, n}(\varphi)$$

Conversely if (*) holds. Then for any $\varphi \in C_0^\infty({}^cK_1)$ $p_{K_1,m,n}(\varphi) = 0$ and hence $|\langle u, \varphi\rangle| \leq c\, p_{K_1,m,n}(\varphi) = 0$

$\Rightarrow u|_{{}^cK_1} = 0$ Hence $\mathrm{Supp}\, u \subset K_1$.

Topology on $\mathcal{E}(\Omega)$: Denote $\mathcal{E}(\Omega) = C^\infty(\Omega)$ and the topology is generated by the family of seminorms $\{p_{K,m,n}\}$. Then $\mathcal{E}(\Omega)$ is a complete metric space. Let $u: \mathcal{E}(\Omega) \to \mathbb{R}(\mathbb{C})$ be a linear map.

Then u is continuous iff $\exists\, c>0$, $K \subset\subset \Omega$ compact and $m \geq 0$ such that $\forall\ \varphi \in \mathcal{E}(\Omega)$

$$|\langle u, \varphi\rangle| \leq c\, p_{K,m,\Omega}(\varphi).$$

Denote $\mathcal{E}'(\Omega)$ = Space of continuous linear maps on $\mathcal{E}(\Omega)$.

Clearly $u \in \mathcal{E}'(\Omega) \Rightarrow u \in \mathcal{D}'(\Omega)$

$u \in \mathcal{E}'(\Omega) \Rightarrow$ supp u is compact.

Let $u \in \mathcal{D}'(\Omega)$ with compact support. Let $K =$ Supp u and let $K \subset K_{2\epsilon} \subset\subset \Omega$. Let $\psi \in C_0^\infty(K_{2\epsilon})$ such that $\psi = 1$ on K_ϵ. Let $\varphi \in C^\infty(\Omega)$. Then

$$\begin{cases} \varphi = \psi\varphi + (1-\psi)\varphi \\ \psi\varphi \in C_0^\infty(K_\epsilon) \end{cases}$$

Define

$$\langle \tilde{u}, \varphi\rangle = \langle u, \psi\varphi\rangle \quad \forall\ \varphi \in \mathcal{E}(\Omega).$$

If $\varphi \in C_0^\infty(\Omega)$, then $(1-\psi)\varphi \in C_0^\infty(\Omega \setminus \bar{K}_\epsilon)$ and hence $\langle u, (1-\psi)\varphi\rangle = 0$. Hence

$$\langle \tilde{u}, \varphi\rangle = \langle u, \varphi\rangle \quad \forall\ \varphi \in C_0^\infty(\Omega)$$

observe that $\tilde{u}$ is independent of the choice of ψ.
Hence we can identify $\mathcal{E}'(\Omega) \subset \mathcal{D}'(\Omega)$ the space of all distributions with compact support.

Lemma:

1. Let $u \in \mathcal{D}'(\Omega)$ be a positive distribution. (i.e. $\langle u, \varphi \rangle \geq 0 \;\; \forall \varphi \in C_0^\infty(\Omega),\ \varphi \geq 0$). Then u is a Radon-measure.

2. Let $u \in \mathcal{D}'(\Omega)$ such that $\operatorname{supp} u = \{x_0\}$. Then $\exists\, m \geq 0$, $c_\alpha \in \mathbb{R}\,(\mathbb{C})$ such that

$$u = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha \delta_{x_0}$$

3. Let $u_{ij} \in \mathcal{D}'(\Omega)$ such that $\forall \varphi \geq 0$
$A(\varphi) = ((\langle u_{ij}, \varphi \rangle))_{1 \leq i,j \leq m} \geq 0$. Then
~~u_{ij} are Radon measures on Ω.~~

4. Let $\Omega_1 \subset\subset \Omega$ be an open set with $\bar{\Omega}_1$ is compact. Let $u \in \mathcal{D}'(\Omega)$, Then $\exists\, f \in L^\infty(\Omega_1)$ and $m \geq 0$ such that $\forall \varphi \in \mathcal{D}(\Omega_1)$

$$\langle u, \varphi \rangle = \langle \partial_1^m \cdots \partial_n^m f, \varphi \rangle.$$

(i.e. locally every distribution is derivatives of an L^∞-function).

Proof (1) Let $K \subset \mathring{K}_1 \subset K_1 \subset \Omega$ be compact sets with $\mathring{K} \neq \phi$. Let $m \geq 0$, $c > 0$ such that $\forall \varphi \in C_0(K_1)$

$$|\langle u, \varphi \rangle| \leq C\, p_{K_1, m, n}(\varphi).$$

Let $0 \leq \Psi \in C_0^\infty(K_1)$ such that $\Psi = 1$ on K.

P.T.O.

Let $\varphi \in C_0^\infty(K)$ with values in $\mathbb{R}$. Let

$$\varphi_\pm = \|\varphi\|_\infty \psi \pm \varphi.$$

Then $\varphi_\pm \geq 0$, $\varphi_\pm \in C_0^\infty(K_1)$. Hence $\langle u, \varphi_\pm\rangle \geq 0$

$$\Rightarrow \pm \langle u, \varphi\rangle \leq \|\varphi\|_\infty \langle u, \psi\rangle$$

$$\Rightarrow \quad |\langle u, \varphi\rangle| \leq |\langle u, \psi\rangle| \, \|\varphi\|_\infty.$$

Hence u is a finite measure on K. Let $K_j \uparrow \Omega$, then $u|_{K_j}$ is a measure $\Rightarrow u$ is a Radon measure on Ω.

(2) In order to prove this, we need a small proposition.

Proposition: Let $R : \Omega \to \mathbb{R}(\mathbb{C})$ be a C^∞ function such that at $x_0 \in \Omega$, $\partial^\alpha R(x_0) = 0 \quad \forall |\alpha| \leq m$.
Then $\exists$ a family of functions $\{R_\epsilon\}$ in $C^\infty(\Omega)$ such that

(i) $R_\epsilon = 0 \quad \forall |x - x_0| < \epsilon$.

(ii) $\sup_\Omega |\partial^\alpha(R_\epsilon - R)| \to 0$ as $\epsilon \to 0$, $\forall |\alpha| \leq m$.

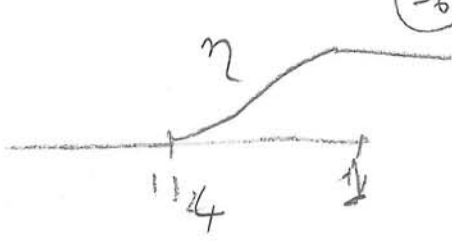
Proof. By Taylor's series, we have for $|x - x_0| < r_0$,

$$R(x) = \sum_{|\alpha| \leq m} \frac{R^{(\alpha)}(x_0)}{\alpha!}(x - x_0)^\alpha + O(|x - x_0|^{m+1})$$

$$\forall |x - x_0| < r_0.$$

Hence $R(x) = O(|x - x_0|^{m+1})$ where $\overline{B(x_0, r_0)} \subset \Omega$.

Let $\eta \in C^\infty(\mathbb{R})$ such that



$$\eta(t) = \begin{cases} 0 & \text{if } t \le \frac{1}{4} \\ 1 & \text{if } t \ge 1 \end{cases}$$

and $\varphi(x) = \eta(|x-x_0|^2) \in C^\infty(\mathbb{R}^n)$ such that

$$\varphi(x) = \begin{cases} 0 & \text{if } |x| \le \frac{1}{2} \\ 1 & \text{if } |x| \ge 1 \end{cases}$$

Let $0<\epsilon<r_0$ and $\varphi_\epsilon(x) = \varphi(x/\epsilon)$. Then for $r\neq 0$

$$\varphi_\epsilon(x) = \begin{cases} 0 & \text{if } |x-x_0| \le \epsilon/2 \\ 1 & \text{if } |x-x_0| \ge \epsilon. \end{cases}$$

Let

$$\begin{cases} R_\epsilon(x) = \varphi_\epsilon(x) R(x) \in C^\infty(\Omega) \\ R_\epsilon(x) = \begin{cases} 0 & \text{if } |x-x_0| \le \epsilon/2 \\ R(x) & \text{if } |x-x_0| > \epsilon. \end{cases} \end{cases}$$

Let $|\alpha| \le m$, then

$$\begin{aligned} \partial^\alpha(R - R_\epsilon) &= \partial^\alpha R - \partial^\alpha(\varphi_\epsilon R) \\ &= \partial^\alpha R - \varphi_\epsilon \partial^\alpha R - \sum_{\beta \le \alpha} c_{\alpha\beta}\, \partial^\beta \varphi_\epsilon\, \partial^{\alpha-\beta} R. \\ &= (1-\varphi_\epsilon)\partial^\alpha R - \sum_{\beta \le \alpha} c_{\alpha\beta}\, \partial^\beta \varphi_\epsilon\, \partial^{\alpha-\beta} R \end{aligned}$$

Now

$$\begin{aligned} (1-\varphi_\epsilon) &= 0 \quad \text{if } |x-x_0| < \epsilon/2 \text{ or } |x-x_0| \ge \epsilon. \\ \partial^\beta \varphi_\epsilon &= 0 \quad \text{if } \beta \neq 0,\ |x-x_0| < \epsilon/2 \text{ or } |x-x_0| > \epsilon. \end{aligned}$$

$$|\partial^{\alpha-\beta} R| = O\left(|x-x_0|^{m+1-|\alpha|+|\beta|}\right) \quad \text{if } |x-x_0| \le r_0.$$

$$\partial^\beta \varphi_\epsilon(x) = O\left(\frac{1}{\epsilon^{|\beta|}}\right) \quad \text{if } \frac{\epsilon}{2} \le |x-x_0| \le \epsilon.$$

Hence for $K_\epsilon: \frac{\epsilon}{2} \le |x-x_0| \le \epsilon$.

$$|\partial^\alpha (R-R_\epsilon)(x)| \le C\Big\{ \sup_{K_\epsilon} |\partial^\alpha R| + \sum_{|\beta|\le|\alpha|} \frac{\epsilon^{m+1-|\alpha|+|\beta|}}{\epsilon^{|\beta|}} \Big\}$$

$$\le C\,\epsilon^{m+1-|\alpha|} \to 0 \text{ as } \epsilon \to 0 \quad \forall\, |\alpha| \le m.$$

This proves the proposition.

Now using the proposition we will prove (2).
Let $u \in \mathcal{D}'(\Omega)$ s.t. $\operatorname{supp} u = \{x_0\}$. Hence $\forall\, \varphi \in C^\infty(\Omega)$ with $\operatorname{supp}\varphi \subset \Omega \setminus \overline{B(x_0,\epsilon)}$, $\langle u, \varphi\rangle = 0$. Let $K = \overline{B(x_0, r_0)} \subset \Omega$ and $0 < \epsilon < r_0$. Then $\exists\, m > 0$ such that $\forall\, \varphi \in C^\infty(\Omega)$

$$|\langle u, \varphi\rangle| \le C\, p_{K,m,\Omega}(\varphi).$$

Let

$$R(x) = \varphi(x) - \sum_{|\alpha|\le m} \frac{\partial^\alpha\varphi(x_0)}{\alpha!}(x-x_0)^\alpha$$

Then $R \in C^\infty(\Omega)$, $\partial^\alpha R(x_0) = 0 \ \forall\, |\alpha| \le m$. Let R_ϵ be as in the above proposition. Since $\operatorname{supp} R_\epsilon \subset \Omega \setminus \overline{B(x_0,\epsilon)}$, hence $\langle u, R_\epsilon\rangle = 0$. Hence

$$|\langle u, R\rangle| = |\langle u, R-R_\epsilon\rangle| \le C\, p_{K,m,\Omega}(R-R_\epsilon) \to 0$$
as $\epsilon \to 0$.

This implies that $\langle u, R\rangle = 0$.

Therefore

$$\langle u, \varphi\rangle = \sum_{|\alpha|\le m} \langle u, \frac{(x-x_0)^\alpha}{\alpha!}\rangle \partial^\alpha \varphi(x_0) + \langle u, R\rangle$$

$$= \sum_{|\alpha|\le m} C_\alpha \partial^\alpha \varphi(x_0)$$

where $C_\alpha = \langle u, \frac{(x-x_0)^\alpha}{\alpha!}\rangle$. This proves (2).

(3) First recall Cauchy-Schwartz inequality:
i.e. let $B: \mathbb{R}^n \to \mathbb{R}^n$ be a symmetric bilinear non negative map. Then $\forall\, x, y \in \mathbb{R}^n$

$$|B(x,y)| \le \sqrt{B(x,x)}\sqrt{B(y,y)}.$$

For $\lambda \in \mathbb{R}$, we have

$$0 \le B(x+\lambda y, x+\lambda y) = B(x,x) + 2\lambda B(x,y) + \lambda^2 B(y,y).$$

Hence the $4B^2(x,y) - 4B(x,x)B(y,y) \le 0$ and this gives the inequality

Let $\varphi \ge 0$ and $B = ((\langle \mu_{ij}, \varphi\rangle))$. Since $B \ge 0$ by hypothesis $\Rightarrow \langle \mu_{ii}, \varphi\rangle \ge 0$ $\forall\, i$ and by Cauchy Schwartz

$$|\langle \mu_{ij}, \varphi\rangle| \le \sqrt{\langle \mu_{ii}, \varphi\rangle}\sqrt{\langle \mu_{jj}, \varphi\rangle}$$

Since $\mu_{ii} \ge 0$ and hence $\{\mu_{ii}\}$ are measures. Let $K \subset \Omega$ be a compact set, then $\exists\, C(K) > 0$ s.t

$$|\langle \mu_{ii}, \varphi\rangle| \le C(K)\|\varphi\|_\infty \quad \forall\, \varphi \in C_0^\infty(K)$$

Hence if $\varphi \geq 0$, then

$$|\langle \mu_{ij}, \varphi\rangle| \leq \sqrt{\langle \mu_{ii}, \varphi\rangle}\sqrt{\langle \mu_{jj}, \varphi\rangle}$$

$$\leq C(K)\,\|\varphi\|_\infty .$$

Now let $0 \leq \psi = 1$ on K, $\psi \in C_0^\infty(\Omega)$, and for $\varphi \in C_0^\infty(K)$

$$\varphi_1 = \|\varphi\|_\infty \psi - \varphi$$

then $\varphi_1 \geq 0$ and hence $\exists\ C(\psi) > 0$ s.t.

$$|\langle \mu_{ij}, \varphi_1\rangle| \leq \sqrt{\langle \mu_{ii}, \varphi_1\rangle}\ \sqrt{\langle \mu_{jj}, \varphi_1\rangle}$$

$$\leq C(\psi)\,\|\varphi\|_\infty .$$

$$|\langle \mu_{ij}, \varphi\rangle| = |\langle \mu_{ij}, \|\varphi\|_\infty \psi - \varphi_1\rangle|$$

$$\leq \|\varphi\|_\infty\, |\langle \mu_{ij}, \psi\rangle| + |\langle \mu_{ij}, \varphi_1\rangle|$$

$$\leq C_1(\psi)\,\|\varphi\|_\infty .$$

Thus proven μ_{ij} are Radon measures.

(ii) Let $\varphi \in C_0^\infty(\Omega) \subset C_0^\infty(\mathbb{R}^n)$. $\alpha = (\alpha_1 \cdots \alpha_n)$, $m > |\alpha| + 1$.
Then by integration by part.

$$\partial^\alpha \varphi(x) = C_{m\alpha} \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} (y_1 - x_1)^{m-\alpha_1-1} \cdots (y_n - x_n)^{m-\alpha_n-1}\, \partial_1^m \cdots \partial_n^m \varphi .$$

where
$$C_{m,\alpha} = \frac{(-1)^{|\alpha|}}{\prod_{j=1}^{m-\alpha_1+1}(m-\alpha_1-j)\prod_{j=1}^{m-\alpha_2+1}(m-\alpha_2-j)\cdots\prod_{j=1}^{m-\alpha_n+1}(m-\alpha_n-j)} .$$

Let $\operatorname{Supp}\varphi \subset \prod_{i=1}^{n} [-R, R] = Q(R)$

Then $\forall y, x \in Q(R)$, $|\alpha| \le m-1$, $|y_i - x_i| \le R$.

$$|\partial^\alpha \varphi| \le C \int_{\mathbb{R}^n} |\partial_1^m \cdots \partial_n^m \varphi|$$

$C = C(R, m, \alpha)$. Hence for $L+1 < m$, we have

$$\sum_{|\alpha| \le L} \|\partial^\alpha \varphi\|_\infty \le C \int_{\mathbb{R}^n} |\partial_1^m \cdots \partial_n^m \varphi|$$

But $\varphi \in C_0^\infty(\Omega_1)$, then $\exists\ C > 0,\ L \ge 0$ s.t.

$$|\langle u, \varphi \rangle| \le C \sum_{|\alpha| \le L} \|\partial^\alpha \varphi\|_\infty \le C \int_{\mathbb{R}^n} |\partial_1^m \cdots \partial_n^m \varphi|$$

Let $W = \{ \partial_1^m \cdots \partial_n^m \varphi : \varphi \in C_0^\infty(\Omega_1) \} \subset L^1(\Omega_1)$

Define the linear map $T : W \to \mathbb{R} / \mathbb{C}$ by

$$\langle T, \partial_1^m \cdots \partial_m^m \varphi \rangle = \langle u, \varphi \rangle \ ?.$$

Then

$$\begin{aligned} |\langle T, \partial_1^m \cdots \partial_n^m \varphi \rangle| &= |\langle u, \varphi \rangle| \\ &\le C \sum_{|\alpha| \le L} \|\partial^\alpha \varphi\|_\infty \\ &\le C \int_{\mathbb{R}^n} |\partial_1^m \cdots \partial_n^m \varphi| \end{aligned}$$

Hence T is cont. on W w.r.t. L^1 norm. By Hahn-Banach theorem, $\exists\ \tilde{T} : L^1(\Omega_1) \to \mathbb{R}\ (\mathbb{C})$ s.t. $\tilde{T}|_W = T$ and $\tilde{T}$ is bounded. Hence by Riesz representation then $\exists\, g \in L^\infty(\Omega_1)$ s.t. $\forall f \in L^1(\Omega_1)$

$$\langle \tilde{T}, f\rangle = \int_{\mathbb{R}^n} g f \, dx.$$

Hence if $f = \partial_1^m \cdots \partial_n^m \varphi \in W$, then

$$\langle u, \varphi\rangle = \langle \tilde{T}, f\rangle = \int_{\mathbb{R}^n} g\, \partial_1^m \cdots \partial_n^m \varphi \, dx.$$

i.e. $\quad u = (-1)^{nm} \partial_1^m \cdots \partial_n^m g$. in Ω_1

This proves (4) and hence the Lemma.

<u>Convolution of two distributions</u>: Let $u_1, u_2 \in \mathcal{D}'(\mathbb{R}^n)$ such that one of them is having compact support, say u_1. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$. Then

<u>Claim</u>: $\operatorname{Supp}(u_1 * \varphi) \subset \operatorname{Supp} u_1 + \operatorname{Supp} \varphi$.

For if $x \notin \operatorname{Supp} u_1 + \operatorname{Supp}\varphi \Rightarrow x - y \notin \operatorname{Supp} u_1$ $\forall$ $y \in \operatorname{Supp}\varphi$. Hence $\operatorname{Supp}(\tau_x \varphi) \subset \mathbb{R}^n \setminus \operatorname{Supp} u_1 \Rightarrow$ $(u_1 * \varphi)(x) = \langle u_1, \tau_x \varphi\rangle = 0$. This proves the claim.

<u>Observation</u>:

(i) $u_1 * \varphi \in C^\infty(\mathbb{R}^n)$

(ii) $\operatorname{Supp}(u_1 * \varphi) \subset \operatorname{Supp} u_1 + \operatorname{Supp}\varphi \to$ compact.

Hence $u_1 * \varphi \in C_0^\infty(\mathbb{R}^n) \Rightarrow \langle u_2, \tau_x (u_1 * \varphi)\rangle$ is well defined.

(iii) $u_2 * \varphi \in C^\infty$, $u_1 \in \mathcal{E}'(\mathbb{R}^n)$, hence

$\langle u_1, \tau_x (u_2 * \varphi)\rangle$ is well defined.

Remum:

$$\langle u_1, \tau_x(u_2 * \varphi)\rangle = \langle u_2, \tau_x(u_1 * \varphi)\rangle$$

Proof: Let u_1 be a function, then

$$\langle u_1, \tau_x(u_2 * \varphi)\rangle = u_1 * (u_2 * \varphi)(x)$$

$$= \int u_1(x-y)\,(u_2 * \varphi)(y)\,dy$$

$$= \int u_1(x-y)\,\langle u_2, \tau_y\varphi\rangle\,dy.$$

$$= \int \langle u_2\ u_1(x-y)\,\tau_y\varphi\rangle\,dy$$

By Rieman approximation as we did earlier to obtain

$$= \langle u_2 \int u_1(x-y)\,\tau_y\varphi\,dy\rangle$$

$$= \langle u_2 \int u_1(x-y)\,\varphi(z-y)\,dy\rangle$$

$$= \langle u_2 \int u_1(x+z-y)\,\varphi(y)\,dy\rangle$$

$$= \langle u_2\ \tau_x(u_1 * \varphi)\rangle$$

Since $(u_1 * \varphi)(z) = \int u_1(z-y)\,\varphi(y)\,dy.$

$$\tau_x(u_1 * \varphi)(z) = (u_1 * \varphi)(x-z)$$

$$= \int u_1(x-z-y)\,\varphi(y)\,dy.$$

Now let $u_1 \in \mathcal{E}'(\mathbb{R}^n)$, $u_{1\epsilon} = \rho_\epsilon * u_1$, $\{\rho_\epsilon\}$ – Mollifiers,

Then

$$\langle u_{1\epsilon}, \tau_x(u_2 * \varphi)\rangle = \langle u_2\ \tau_x(u_{1\epsilon} * \varphi)\rangle$$

letting $\epsilon \to 0$ to obtain

$$\langle u_1, \tau_x(u_2 * \varphi)\rangle = \langle u_2, \tau_x(u_1 * \varphi)\rangle.$$

This proves the Lemma.

Let $f_1, f_2, f_3 \in C_0^\infty(\mathbb{R}^n)$, then

$$\langle f_1 * f_2, f_3\rangle = \int (f_1 * f_2)(y) f_3(y)\, dy$$

$$= \iint f_1(z) f_2(y-z) f_3(y)\, dy\, dz$$

$$= \int f_1(z) (\check{f}_2 * f_3)(z)\, dz$$

$$= \langle f_1, \check{f}_2 * f_3\rangle$$

<u>Definition</u>: Let $u_1 \in \mathcal{E}'(\mathbb{R}^n)$, $u \in \mathcal{D}'(\mathbb{R}^n)$, $\varphi \in C_0^\infty(\mathbb{R}^n)$ define

$$\langle u_1 * u_2, \varphi\rangle = \langle u_1, \check{u}_2 * \varphi\rangle = \langle u_2, \check{u}_1 * \varphi\rangle.$$

<u>Theorem</u>: $\forall\, u \in \mathcal{D}'$, $u * \delta_0 = u$.

<u>Proof</u>: $\langle u * \delta_0, \varphi\rangle = \langle u, \check{\delta}_0 * \varphi\rangle$.

Then $(\check{\delta}_0 * \varphi)(x) = \langle \delta_0, \tau_x \varphi\rangle = \check{\tau_x\varphi}(0) = \ldots$

$(\tau_x\varphi)^\vee(0) = (\tau_x \varphi)(-0) = \varphi(x)$

Hence $\check{\delta}_0 * \varphi = \delta * \varphi = \varphi(x)$.

Fourier Transform.

Tempered functions $\mathcal{S}$: Let $\varphi \in C^\infty(\mathbb{R}^n)$, $m, k \geq 0$. Define

$$p_{m,k}(\varphi) = \sup\Big[(1+|x|^2)^m \sum_{|\alpha| \leq k} |\partial^\alpha \varphi(x)|\Big].$$

Then $\mathcal{S}$ is defined by

$$\mathcal{S} = \{\varphi \in C^\infty(\mathbb{R}^n) : p_{m,k}(\varphi) < \infty \ \forall\, m, k\}.$$

(i) Clearly, $C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}$.

(ii) $\varphi(x) = e^{-|x|^2} \in \mathcal{S}$.

Properties:

(1) $\mathcal{S}$ is a vector space

(2) $\varphi, \psi \in \mathcal{S} \Rightarrow \varphi\psi \in \mathcal{S}$.

(3) $\mathcal{S}$ w.r.t. the family of seminorms $\{p_{m,k}\}$ form a complete metric space.

(4) $\varphi \in \mathcal{S} \Rightarrow \forall\, \alpha, \beta$, $x^\alpha \partial^\beta \varphi \in \mathcal{S}$.

(5) $\varphi \in \mathcal{S} \Rightarrow \varphi \in L^p(\mathbb{R}^n) \ \forall\, 1 \leq p \leq \infty$.

(6) $\varphi, \psi \in \mathcal{S}$, then $\varphi * \psi \in \mathcal{S}$.

Proof: (1) and (2) are obvious.

Let $d(\varphi, \psi) = \displaystyle\sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{2^{m+k}} \frac{p_{m,k}(\varphi - \psi)}{1 + p_{m,k}(\varphi - \psi)}$

is a metric. $\{\varphi_k\}$ is a cauchy sequence in d iff

$p_{m,k}(\varphi_l - \varphi_{l'}) \to 0$ as $l, l' \to \infty$ $\forall$ m, k. Hence by taking $m = 0$, it follows that $\forall\, \alpha$

$$\|\partial^\alpha \varphi_l - \partial^\alpha \varphi_{l'}\|_\infty \to 0 \quad \text{as } l, l' \to \infty.$$

Hence $\lim_{l\to\infty} \varphi_l = \varphi \in C^\infty(\mathbb{R}^n)$. and $p_{m,k}(\varphi_l - \varphi) \to 0$ as $l \to \infty \Rightarrow p_{m,k}(\varphi) < \infty$ $\forall$ $m, k \Rightarrow \varphi \in \mathcal{S}$

(4)

$$p_{m,k}(x^\alpha \partial^\beta \varphi) = \sup (1+|x|)^m \sum_{|\gamma|\le k} |\partial^\gamma (x^\alpha \partial^\beta \varphi)|$$

$$\le C \sup\Big((1+|x|)^{m+|\alpha|} \sum_{|\eta| \le k+|\beta|} |\partial^\beta \varphi|\Big)$$

$$= C\, p_{m+|\alpha|,\, k+|\beta|}(\varphi) < \infty.$$

(5) $\forall\, m \ge 0$, $(1+|x|)^m |\varphi(x)| \le p_{m,0}(\varphi) < \infty$.

Hence $|\varphi(x)| \le \dfrac{p_{m,0}(\varphi)}{(1+|x|)^m}$.

$$\Rightarrow \int |\varphi(x)|^p\, dx \le p_{m,0}(\varphi)^p \int \frac{dx}{(1+|x|)^{mp}} < \infty \quad \text{if } m > n/p.$$

(6) $|x| = |x-y+y| \le |x-y| + |y|$

$$1+|x| \le 1+|x-y|+|y| \le (1+|x-y|)(1+|y|)$$

$$(1+|x|)^m \le (1+|x-y|)^m (1+|y|)^m.$$

Hence

$$\left|(1+|x|)^m (\varphi * \psi)(x)\right| = \left|\int (1+|x|)^m \varphi(x-y)\psi(y)\,dy\right|$$

$$\leq \int (1+|x-y|)^m |\varphi(x-y)| \, (1+|y|)^m |\psi(y)|\,dy$$

$$\leq p_{m,0}(\varphi) \int (1+|y|)^m |\psi(y)|\,dy$$

$$\leq p_{m,0}(\varphi) \int \frac{(1+|y|)^m}{(1+|y|)^{m+n+1}} (1+|y|)^{m+n+1} |\psi(y)|\,dy$$

$$\leq p_{m,0}(\varphi)\, p_{m+n+1}(\psi) \int \frac{dy}{(1+|y|)^{n+1}}$$

$$\leq c\, p_{m,0}(\varphi)\, p_{m+n+1}(\psi)$$

Now

$$\left|(1+|x|)^m \partial^\alpha(\varphi * \psi)\right| = \left|(1+|x|)^m \partial^\alpha\varphi * \psi\right|$$

$$\leq c\, p_{m,0}(\partial^\alpha \varphi)\, p_{m+n+1}(\psi)$$

$$\leq c\, p_{m,|\alpha|}(\varphi)\, p_{m+n+1}(\psi).$$

Fourier Transform.

For $f \in L^1(\mathbb{R}^n)$, define

$$\hat{f}(\xi) = \int e^{-i\langle x,\xi\rangle} f(x)\,dx.$$

called the fourier transform of f.

Riemann-Lebesgue Lemma: $f \in L^1(\mathbb{R}^n)$, Then $\hat{f} \in C^0(\mathbb{R}^n)$ and $|\hat{f}(\xi)| \to 0$ as $|\xi| \to \infty$.

Proof: $C_0^\infty(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$ is dense

$$(1-\Delta)\, e^{-i\langle x,\xi\rangle} = (1+|\xi|^2)\, e^{-i\langle x,\xi\rangle}$$

Hence for $\varphi \in C_0^\infty(\mathbb{R}^n)$

$$\hat{\varphi}(\xi) = \int e^{-i\langle x,\xi\rangle} \varphi(x)\, dx$$

$$= \int \frac{(1+|\xi|^2)}{(1+|\xi|^2)} e^{-i\langle x,\xi\rangle} \varphi(x)\, dx$$

$$= \frac{1}{(1+|\xi|^2)} \int (1-\Delta)\, e^{-i\langle x,\xi\rangle} \varphi\, dx$$

$$= \frac{1}{(1+|\xi|^2)} \int e^{-i\langle x,\xi\rangle} (1-\Delta)\varphi\, dx$$

$\to 0$ as $|\xi| \to \infty$.

Let $f \in L^1$, and $\epsilon > 0$. choose $\varphi \in C_0^\infty(\mathbb{R}^n)$ s.t. $\|f-\varphi\|_{L^1} < \epsilon$

$$\hat{f}(\xi) = \int e^{-i\langle x,\xi\rangle} f(x)\, dx$$

$$= \int e^{-i\langle x,\xi\rangle} (f-\varphi)\, dx + \hat{\varphi}(\xi)$$

$$|\hat{f}(\xi)| \leq \|f-\varphi\|_{L^1} + |\hat{\varphi}(\xi)| < \epsilon + |\hat{\varphi}(\xi)|$$

letting $\xi \to \infty$, $\epsilon \to 0$ to obtain $|\hat{f}(\xi)| \to 0$ as $|\xi| \to \infty$.

Properties of Fourier Transform.

(1) $f_1, f_2 \in L^1(\mathbb{R}^n)$. Then by Fubini

$$\int_{\mathbb{R}^n} f_1(x)\hat{f}_2(x) e^{i\langle x,\xi\rangle} dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f_1(x) f_2(y) e^{-i\langle y,x\rangle + i\langle x,\xi\rangle} dy\, dx$$

$$= \int_{\mathbb{R}^n} f_2(y)\left(\int f_1(x) e^{-i\langle x, y-\xi\rangle} dx\right) dy$$

$$= \int f_2(y)\hat{f}_1(y-\xi)\, dy.$$

(2) Let $\lambda > 0$ and for $f \in L^1(\mathbb{R}^n)$ define

$$f_\lambda(x) = \lambda^{-n} f(x/\lambda)$$

Then

$$\hat{f}_\lambda(\xi) = \lambda^{-n}\int_{\mathbb{R}^n} f(x/\lambda) e^{-i\langle x,\xi\rangle} dx$$

$$= \int_{\mathbb{R}^n} f(x) e^{-i\langle x,\lambda\xi\rangle} dx$$

$$= \hat{f}(\lambda\xi).$$

(3) Let f be such that $(1+|x|)^m f \in L^1$ for some $m \geq 0$, Then $\hat{f} \in C^m$ and for $|\alpha| \leq m$

$$\partial^\alpha \hat{f}(\xi) = (-i)^{|\alpha|}\int e^{-i\langle x,\xi\rangle} x^\alpha f(x) dx = (-i)^{|\alpha|}(x^\alpha f)^\wedge(\xi).$$

This follows from dominated Convergence Theorem.

(4) Let $f \in C^m(\mathbb{R}^n)$ with $\partial^\alpha f \in L^1$ $\forall\ |\alpha| \leq m$ and

$\partial^\alpha f(\xi) \to 0$ as $|\xi| \to \infty$. Then

$$(\partial^\alpha f)^\wedge(\xi) = (i)^{|\alpha|}\xi^\alpha \hat{f}(\xi)$$

$$\int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}\frac{\partial f}{\partial x_k}dx = \int_{\mathbb{R}^{n-1}} e^{-i\sum_{l\neq k}x_l\xi_l}\left(\int_{\mathbb{R}} e^{-ix_k\xi_k}\frac{\partial f}{\partial x_k}dx_k\right) dx_1 \ldots dx_{k-1}dx_{k+1}\ldots dx_n$$

$$\int_{\mathbb{R}} e^{-ix_k\xi_k}\frac{\partial f}{\partial x_k}dx = \lim_{R\to\infty}\int_{-R}^{R} e^{-ix_k\xi_k}\frac{\partial f}{\partial x_k}dx_k$$

$$= \lim_{R\to\infty} e^{-ix_k\xi_k}\Big[f(x_1 \ldots x_{k-1}, R, x_{k+1} \ldots x_n) - f(x_1 \ldots x_{k-1}, -R, x_{k+1} \ldots x_n)\Big]$$

$$- \lim_{R\to\infty}\int_{-R}^{R}(-i)\xi_k e^{-ix_k\xi_k} f\, dx_k$$

$$= i\xi_k\int_{\mathbb{R}} e^{-ix_k\xi_k} f\, dx_k.$$

$$\therefore \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}\frac{\partial f}{\partial x_k}dx = i\xi_k\int e^{-i\langle x,\xi\rangle} f(x)\,dx$$

$$= i\xi_k\hat{f}(\xi)$$

By induction it follows.

(5) Explicit Computations:

(i) Let $f(x) = \chi_{[a,b]}(x)$. Then

$$\hat{f}(\xi) = \int_a^b e^{-ix\xi}dx = \frac{1}{-i\xi}e^{-ix\xi}\Big]_a^b$$

$$= i\,\frac{e^{-ib\xi} - e^{-ia\xi}}{\xi}.$$

In particular if $a = -b$, Then

$$\hat{f}(\xi) = \frac{i}{\xi}\left(e^{-ib\xi} - e^{ib\xi}\right) = -\frac{i^2}{\xi}\sin b\xi$$

$$= \frac{\sin b\xi}{\xi} \notin L^1(\mathbb{R})$$

(ii) $f(x) = \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{if } x < 0. \end{cases}$

$$\hat{f}(\xi) = \int_0^\infty e^{-ix\xi - x}\,dx = \int_0^\infty e^{-x(1+i\xi)}\,dx$$

$$= -\frac{1}{(1+i\xi)} e^{-x(1+i\xi)}\Big]_0^\infty$$

$$= \frac{1}{1+i\xi} \notin L^1(\mathbb{R}).$$

(iii) $|x|^2 + i\langle x,\xi\rangle = \sum_{j=1}^{n}(x_j^2 + ix_j\xi_j)$

$$= \sum_{j=1}^{n}\left(x_j + \frac{i}{2}\xi_j\right)^2 + \frac{1}{4}\sum_{j=1}^{n}\xi_j^2.$$

$$= \langle x + \tfrac{i}{2}\xi, x + \tfrac{i}{2}\xi\rangle + \frac{1}{4}|\xi|^2.$$

Put $f(x) = e^{-|x|^2}$, Then

$$\hat{f}(\xi) = \int e^{-|x|^2 - i\langle x,\xi\rangle}\,dx$$

$$= e^{-\frac{1}{4}|\xi|^2}\int_{\mathbb{R}^n} e^{-\langle x + \frac{i\xi}{2},\, x + i\xi/2\rangle}\,dx$$

$$\hat{f}(\xi) = e^{-\frac{1}{4}|\xi|^2}\int_{\mathbb{R}^n} e^{-|x|^2}\,dx.$$

Let $I = \int_0^\infty e^{-t^2} dt.$

Then $I^2 = \int_0^\infty \int_0^\infty e^{-t^2-s^2} ds\, dt$

$$= \int_{\substack{x_1>0 \\ x_2>0}} e^{-|x|^2} dx$$

$$= \int_0^\infty \int_0^{\pi/2} e^{-r^2} r\, dr\, d\theta$$

$$= \frac{\pi}{4} \int_0^\infty e^{-r^2} d(r^2) = \frac{\pi}{4}.$$

Hence $I = \frac{\sqrt{\pi}}{2}$. Hence

$$\int_{-\infty}^\infty e^{-t^2} dt = 2\int_0^\infty e^{-t^2} dt = \sqrt{\pi}.$$

$$\therefore \int_{\mathbb{R}^n} e^{-|x|^2} dx = \left(\int_{\mathbb{R}} e^{-t^2} dt\right)^n = \pi^{n/2}.$$

Hence

$$\boxed{\hat{f}(\xi) = \pi^{n/2} e^{-|\xi|^2/4}.}$$

Let $f_a(x) = e^{-a|x|^2}$, then $\hat{f_a}(\xi) = \int e^{-a|x|^2} e^{-i\langle x, \xi\rangle} dx$

$$= \frac{1}{a^{n/2}} \int e^{-|x|^2 - i\langle x, \frac{\xi}{\sqrt{a}}\rangle}$$

$$= \frac{1}{a^{n/2}} \hat{f}(\xi/\sqrt{a})$$

$$= \left(\frac{\pi}{a}\right)^{n/2} e^{-\frac{|\xi|^2}{a}}.$$

(6) $f_1, f_2 \in L^1(\mathbb{R}^n)$, then
$$(f_1 * f_2)^\wedge = \hat{f}_1(\xi)\hat{f}_2(\xi).$$

$$(f_1 * f_2)^\wedge(\xi) = \int \bar{e}^{i\langle x,\xi\rangle}\left(\int f_1(x-y) f_2(y)\, dy\right) dx$$
Then by Fubini,
$$= \int f_2(y)\left(\int f_1(x-y)\, \bar{e}^{i\langle x,\xi\rangle}\, dx\right) dy$$
$$= \int f_2(y)\, \bar{e}^{i\langle y,\xi\rangle}\left(\int f_1(x)\, \bar{e}^{i\langle x,\xi\rangle}\, dx\right) dy$$
$$= \hat{f}_1(\xi)\hat{f}_2(\xi).$$

Fourier Inversion and Plancheral formula.

Let $f \in L^1 \cap C^0$, L^1 such that $\hat{f} \in L^1$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then from (1) we have
$$\int \varphi(x)\hat{f}(x) e^{i\langle x,\xi\rangle} = \int f(y)\hat{\varphi}(y-\xi)\, dy.$$
Let $\varphi_\epsilon(x) = \varphi(\epsilon x) \Rightarrow \hat{\varphi}_\epsilon(\xi) = \frac{1}{\epsilon^n}\hat{\varphi}(\xi/\epsilon)$.
$$\therefore \int \varphi(\epsilon x)\hat{f}(x)\bar{e}^{i\langle x,\xi\rangle} = \int f(y)\,\hat{\varphi}\left(\frac{(y-\xi)}{\epsilon}\right)\frac{dy}{\epsilon^n}$$
$$\int \varphi(\epsilon x)\hat{f}(x) e^{i\langle x,\xi\rangle} dx = \int \hat{\varphi}(y) f(\xi+\epsilon y)\, dy$$
Since $f, \hat{f} \in L^1 \cap C^0$, hence by dominated convergence theorem, letting $\epsilon \to 0$ to obtain
$$\varphi(0)\int \hat{f}(x) e^{i\langle x,\xi\rangle} dx = f(\xi)\int \hat{\varphi}(y)\, dy.$$

Hence if $\varphi(0) \neq 0$, Then

$$f(\xi) = \left(\frac{\varphi(0)}{\int \hat{\varphi}(y)\,dy}\right) \int \hat{f}(x) e^{i\langle x,\xi\rangle} dx.$$

Now let $\varphi(x) = e^{-|x|^2}$, Then $\hat{\varphi}(\xi) = \pi^{n/2} e^{-|\xi|^2/4}$.

Hence

$$\int \hat{\varphi}(\xi) = \pi^{n/2} \int e^{-|\xi|^2/4} d\xi = (4\pi)^{n/2} \int e^{-|\xi|^2} d\xi$$

$$= (4\pi)^{n/2} \pi^{n/2} = (2\pi)^n.$$

Hence we have the Fourier Inversion formula

$$\boxed{f(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(x) e^{i\langle x,\xi\rangle} dx.}$$

Hence

$$f(\xi) = \frac{1}{(2\pi)^n} \hat{\hat{f}}(-\xi) \to (2\pi)^n f(-\xi) = \hat{\hat{f}}(\xi).$$

Parsavalls formula.

Let $f, g \in L^1 \cap C^0$, $\hat{f}, \hat{g} \in L^1$. Then

$$\int_{\mathbb{R}^n} f \hat{g}\, dx = \int_{\mathbb{R}^n} \hat{f} g\, dy.$$

$$\int \hat{f} \overline{\hat{g}}\, dx = \int \hat{f}(x) \left(\overline{\int g(y) e^{-i\langle y,x\rangle} dy}\right) dx$$

$$= \int \overline{g}(y) \left(\int \hat{f}(x) e^{i\langle y,x\rangle} dx\right) dy$$

$$= (2\pi)^n \int \overline{g}(y) f(y)\, dy.$$

i.e.

$$\int f\hat{g} = \int \hat{f} g$$

$$\int \hat{f}\bar{\hat{g}} = (2\pi)^n \int f\bar{g}$$

By taking $f=g$ to obtain

$$\boxed{\int_{\mathbb{R}^n} |\hat{f}|^2 = (2\pi)^n \int_{\mathbb{R}^n} |f|^2}$$ Parasaval's formula.

As a consequence of all these analysis we have the following

<u>Theorem</u> (1) $\wedge : \mathcal{S} \to \mathcal{S}$ is an Isomorphism.

(2) $\mathcal{S} \subset L^2(\mathbb{R}^n)$ is a dense set. Then $\wedge : \mathcal{S} \to \mathcal{S}$ extends as an Isometry for $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

(3) For $f, g \in \mathcal{S}$.

$$\int f\hat{g} = \int \hat{f} g.$$

<u>Tempered distributions $\mathcal{S}'$</u>

$\mathcal{S}'$ = Topological dual of $\mathcal{S}$.

i.e. $u \in \mathcal{S}'$ iff $u : \mathcal{S} \to \mathbb{R}(\mathbb{C})$ is a linear map and $\exists\, c>0, m, k$ s.t. $\forall\, \varphi \in \mathcal{S}$

$$|\langle u, \varphi\rangle| \le c\, p_{m,k}(\varphi)$$

<u>Remark</u>: $\mathcal{S}' \subset \mathcal{D}'$.

For if $K \subset \mathbb{R}^n$ is a compact set and $\varphi \in C^\infty(K)$,

then $p_{m,k}(\varphi) \le c\, p_{K,k,n}(\varphi)$.

Examples:

(1) $L^p(\mathbb{R}^n) \subset \mathcal{S}'$

For $f \in L^p(\mathbb{R}^n)$ Then for $\varphi \in \mathcal{S}$

$$|\langle f, \varphi\rangle| = |\int f\varphi|$$
$$\leq \|f\|_p \|\varphi\|_{L^q} \qquad \frac{1}{p}+\frac{1}{q}=1.$$
$$\leq C\|f\|_p\, p_{m,0}(\varphi) \text{ for suitable } m>0.$$

(2) $\mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'$.

$u \in \mathcal{E}'(\mathbb{R}^n)$ iff $\exists m, C>0, K \subset\subset \mathbb{R}^n$ s.t. $\forall \varphi \in C^\infty(\mathbb{R}^n)$

$$|\langle u, \varphi\rangle| \leq C\, p_{K,m,\mathbb{R}^n}(\varphi)$$

Now

$$p_{K,m,\mathbb{R}^n}(\varphi) = \sup_K \sum_{|\alpha|\leq m} |\partial^\alpha \varphi|. \leq p_{0,m}(\varphi).$$

Hence

$$|\langle u, \varphi\rangle| \leq C\, p_{0,m}(\varphi) .$$

(3) $\mathcal{D}' \not\subset \mathcal{S}'$

For if $f(x) = e^{|x|^2} \in C^\infty(\mathbb{R}^n)$. Then $f \notin \mathcal{S}'$.

Since $e^{-|x|^2/2} \in \mathcal{S}$ and $\int e^{|x|^2} e^{-|x|^2/2} = \infty$

(4) $\partial^\alpha : \mathcal{S}' \longrightarrow \mathcal{S}'$, since $\partial^\alpha : \mathcal{S} \to \mathcal{S}$ is continuous.

$$\langle \partial^\alpha u, \varphi\rangle = (-1)^{|\alpha|}\langle u, \partial^\alpha \varphi\rangle$$

(5) $x^\beta : \mathcal{S}' \longrightarrow \mathcal{S}'$ by,

$$\langle x^\beta u, \varphi\rangle = \langle u, x^\beta \varphi\rangle$$

since $x^\beta : \mathcal{S} \to \mathcal{S}$ is continuous

(6) $u \in \mathcal{S}'$, $f \in \mathcal{S}$ then $u * f \in C^\infty$ and $\exists$ $c > 0$, $m > 0$ s.t.

$$|(u * f)(x)| \le c(1+|x|)^m$$

For

$$|(u * f)(x)| = |\langle u, \tau_x f\rangle|$$

$$\le c\, p_{m,k}(\tau_x f).$$

$$p_{m,k}(\tau_x f) = \sup_y \Big[(1+|y|)^m \sum_{|\alpha|\le k} |\partial_y^\alpha f(x-y)|\Big]$$

$$= \sup_y \Big[(1+|x+y|)^m \sum_{|\alpha|\le k} |\partial^\alpha f(y)|\Big]$$

$$\le (1+|x|)^m \sup_y (1+|y|)^m \sum_{|\alpha|\le k} |\partial^\alpha f(y)|$$

$$= c(1+|x|)^m.$$

Hence $u * f$ is of polynomial growth.

(7) If $g \in C^\infty$ function such that $\forall\, \alpha$ $\partial^\alpha g$ is of polynomial growth, then $gu \in \mathcal{S}'$ if $u \in \mathcal{S}'$. Since $f \in \mathcal{S} \Rightarrow gf \in \mathcal{S}$ and

$$\langle gu, f\rangle = \langle u, gf\rangle$$

(8) Let $u \in \mathcal{E}'$, $f \in \mathcal{S}$. Then $u * f \in \mathcal{S}$.

$$|(1+|x|)^m (u * f)(x)| = |(1+|x|)^m \langle u, \tau_x f\rangle|$$

$$\le (1+|x|)^m \sum_{|\alpha|\le m} \sup_{y\in K} |\partial^\alpha f(x-y)|$$

Since $f \in \mathcal{S} \Rightarrow \forall\, \alpha, \ell$

$$(1+|z|)^\ell |\partial^\alpha f(z)| \le C_{\ell,\alpha}.$$

Hence

$$|\partial^\alpha f(x-y)| \le \frac{C_{\alpha,l}}{(1+|x-y|)^l}$$

$$\sup_{y\in K}|\partial^\alpha f(x-y)| \le \sup_{y\in K}\frac{C_{\alpha,l}}{(1+|x-y|)^l} \le \frac{\tilde{C}_{\alpha,l}}{(1+|x|)^l}.$$

Hence

$$|(1+|x|)^m (u*f)(x)| \le C_{l,}(1+|x|)^{m-l}.$$

Now choose $l=m$ to obtain

$$|(1+|x|)^m (u*f)(x)| \le C_m$$

Since $\partial^\alpha f \in \mathcal{Y} \Rightarrow$

$$|(1+|x|)^m (u*\partial^\alpha f)(x)| \le C_{m,\alpha}$$

$\Longrightarrow$ $u*f \in \mathcal{Y}$.

Fourier Transform of Tempered distributions.

$\wedge : \mathcal{Y}\to\mathcal{Y}$ is an Isomorphism.

Hence by duality

$\wedge : \mathcal{Y}'\to\mathcal{Y}'$ is an Isomorphism given by

$$\langle \hat{u}, f\rangle = \langle u, \hat{f}\rangle \quad \forall u\in\mathcal{Y}',\ f\in\mathcal{Y}.$$

Examples:

(1) $u=\delta_{x_0}$, Then

$$\langle \hat{u}, f\rangle = \langle \delta_{x_0}, \hat{f}\rangle = \hat{f}(x_0) = \int f(y)\, e^{-i\langle x_0, y\rangle} dy$$

Hence $\hat{\delta}_{x_0} = e^{-i\langle x_0, y\rangle}$.

In particular if $x_0=0$, then $\hat{\delta}=1$

(2) $u \in \mathcal{E}'$, Then
$$\hat{u}(\xi) = \langle u, e^{-i\langle \cdot, \xi \rangle} \rangle.$$
For $f \in \mathcal{S} \Rightarrow$
$$\langle \hat{u}, f \rangle = \langle u, \hat{f} \rangle = \langle u_\xi, \int f(x) e^{-i\langle x, \xi \rangle} dx \rangle$$
$$= \int f(x) \langle u_\xi, e^{-i\langle x, \cdot \rangle} \rangle dx$$
(Justified by taking Riemann Sum)

Hence
$$\hat{u}(\xi) = \langle u, e^{-i\langle \cdot, \xi \rangle} \rangle$$

(3) Put $u = x^\alpha$, Then
$$\langle \hat{u}, f \rangle = \langle x^\alpha, \hat{f} \rangle$$
$$= \int x^\alpha \hat{f}(x) dx$$
$$= \frac{1}{(i)^{|\alpha|}} \int (\partial^\alpha f)^\wedge dx$$
$$= \frac{(2\pi)^n}{i^{|\alpha|}} \partial^\alpha f(0)$$
Hence $\widehat{x^\alpha} = C_\alpha \partial^\alpha \delta_0$.

(4) $\widehat{\partial^\alpha u} = (-i)^{|\alpha|} x^\alpha \hat{u}$
$$\langle \widehat{\partial^\alpha u}, f \rangle = (-1)^{|\alpha|} \langle u, \partial^\alpha \hat{f} \rangle$$
$$= (-1)^{|\alpha|} \langle u, (-i)^{|\alpha|} \widehat{x^\alpha f} \rangle$$
$$= (i)^{|\alpha|} \langle \hat{u}, x^\alpha f \rangle$$
$$= (-i)^{|\alpha|} \langle x^\alpha \hat{u}, f \rangle.$$

(5) For $f, \varphi \in \mathcal{S}$, $\epsilon > 0$

$$\int \varphi(\epsilon x)\, \hat{f}(x)\, dx = \int f(y) \frac{1}{\epsilon^n} \hat{\varphi}\left(\frac{y}{\epsilon}\right) dy.$$

Let $\varphi(x) = e^{-|x|^2}$.

$$\hat{\varphi}(\xi) = \pi^{n/2} e^{-|\xi|^2/4}, \quad \hat{\varphi}\left(\frac{\xi}{\sqrt{\epsilon}}\right) = \pi^{n/2} e^{-\frac{|\xi|^2}{4\epsilon^2}}$$

$$\int e^{-\epsilon|x|^2} \hat{f}(x)\, dx = \frac{\pi^{n/2}}{\epsilon^{n/2}} \int f(\xi)\, e^{-\frac{|\xi|^2}{4\epsilon}}\, d\xi.$$

Since, $\displaystyle\int_0^\infty e^{-a\epsilon} \epsilon^{\alpha-1} d\epsilon = \frac{1}{a^\alpha} \int_0^\infty e^{-t} t^\alpha dt.$

Hence

$$\int_0^\infty \epsilon^{\alpha-1} \int_{\mathbb{R}^n} e^{-\epsilon|x|^2} \hat{f}(x)\, dx = \pi^{n/2} \int_0^\infty \frac{\epsilon^{\alpha-1}}{\epsilon^{n/2}} \left(\int_{\mathbb{R}^n} f(\xi)\, e^{-|\xi|^2/4\epsilon} d\xi \right) d\epsilon.$$

Now

$$\int_0^\infty \epsilon^{\alpha-1} \int e^{-\epsilon|x|^2} \hat{f}(x)\, dx = \int_0^\infty \epsilon^{\alpha-1} \frac{d\epsilon}{\epsilon^{n/2}} \int_{\mathbb{R}^n} e^{-|x|^2} \hat{f}\left(\frac{x}{\sqrt{\epsilon}}\right) dx.$$

$$= \int_0^\infty \frac{d\epsilon}{\epsilon^{\frac{n}{2}-\alpha+1}} \int_{\mathbb{R}^n} e^{-|x|^2} \hat{f}(x/\sqrt{\epsilon})\, dx.$$

For $\alpha > 0$,

$$\int_0^1 \int_{\mathbb{R}^n} |\epsilon^{\alpha-1} e^{-\epsilon|x|^2} \hat{f}(x)|\, dx\, d\epsilon < \infty$$

For $\epsilon > 1$, $|\hat{f}(x/\sqrt{\epsilon})| \leq C$

$$\int_{\mathbb{R}^n} \int_1^\infty \frac{e^{-|x|^2}}{\epsilon^{\frac{n}{2}-\alpha+1}} dx < \infty \quad \text{if} \quad \alpha < n/2.$$

Hence for $0 < 2\alpha < n$,

$$\int_0^\infty \epsilon^{\alpha-1} d\epsilon \int_{\mathbb{R}^n} e^{-\epsilon|x|^2} \hat{f}(x)\, dx = \int_{\mathbb{R}^n} \hat{f}(x) \int_0^\infty e^{-\epsilon|x|^2} \epsilon^{\alpha-1} d\epsilon$$

$$= c \int_{\mathbb{R}^n} \frac{\hat{f}(x)}{|x|^{2\alpha}}$$

Ill^{ly}

$$\int_0^\infty \frac{\epsilon^{\alpha-1}}{\epsilon^{n}} \int_{\mathbb{R}^n} f(y)\, e^{-|y|^2/4\epsilon} dy\, d\epsilon$$

$$= \int_{\mathbb{R}^n} f(y) \left(\int_0^\infty \epsilon^{\alpha-1-n/2} e^{-|y|^2/4\epsilon} d\epsilon \right) dy.$$

Put $t = \frac{1}{\epsilon}$.
$dt = -\frac{d\epsilon}{\epsilon^2}$

$$= \int_{\mathbb{R}^n} f(y) \left(\int_0^\infty \left(\frac{1}{t}\right)^{\alpha+1-n/2} e^{-t\frac{|y|^2}{4}} dt \right) dy$$

$$= c \int_{\mathbb{R}^n} \frac{f(y)\, dy}{|y|^{n-2\alpha}}.$$

Hence

$$\int_{\mathbb{R}^n} \frac{1}{|x|^\alpha} \hat{f}(x)\, dx = c \int \frac{f(y)}{|y|^{n-\alpha}} dy.$$

Hence $\left(\frac{1}{|x|^\alpha}\right)^\wedge = \frac{c}{|y|^{n-\alpha}}$!

if $0 < \alpha < n$

Corollary. Let $E \in \mathcal{S}'$ such that
$$-\Delta E = \delta_0.$$
Then $E = c/|x|^{n-2}$ if $n \geq 3$.

Proof: Taking the Fourier transform we obtain
$$|\xi|^2 \hat{E}(\xi) = 1 \quad \text{or} \quad \hat{E}(\xi) = \frac{1}{|\xi|^2}$$
Hence $E(x) = c/|x|^{n-2}$.

Tauberian theorem of Weiner.

Lemma: Let $f \in L^1$ and $\xi_0 \in \mathbb{R}^n$. Then given $\epsilon>0$ $\exists$ nhd V_{ξ_0} of ξ_0 and $h \in L^1$ such that
$$\begin{cases} \hat{h}(\xi) = \hat{f}(\xi_0) - \hat{f}(\xi+\xi_0) \quad \forall\, \xi \in V_0. \\ \|h\|_{L^1} < \epsilon \end{cases}$$

Proof: Let $\psi \in \mathcal{S}$ such that $\operatorname{supp}\hat{\psi} \subset B(0,1)$ and $\hat{\psi}(\xi) = 1$ if $|\xi| \leq 1/2$. Let $\lambda > 0$, define
$$\psi_\lambda(x) = \lambda^{-n}\psi(x/\lambda)$$
then $\hat{\psi}_\lambda(\xi) = \hat{\psi}(\lambda\xi)$ and $\hat{\psi}(\lambda\xi) = 1$ if $|\xi| \leq \frac{1}{2\lambda}$.

Define
$$\hat{h}(\xi) = \hat{f}(\xi_0)\,\hat{\psi}(\lambda\xi) - \hat{f}(\xi+\xi_0)\,\hat{\psi}(\lambda\xi).$$
Since
$$\begin{aligned}(e^{-i\langle x,\xi_0\rangle} f)^\wedge(\xi) &= \int e^{-i\langle x,\xi_0\rangle} f(x)\, e^{-i\langle x,\xi\rangle}\, dx \\ &= \int e^{-i\langle x,\xi_0+\xi\rangle} f(x)\, dx \\ &= \hat{f}(\xi+\xi_0)\end{aligned}$$

We have

$$h(y) = \hat{f}(\xi_0)\psi_\lambda(y) - \int e^{-i\langle x,\xi_0\rangle} f(x)\psi_\lambda(y-x)\,dx.$$

$$\cancel{\int |h(y)|dy} = \psi_\lambda(y)\int e^{-i\langle x,\xi_0\rangle} f(x)\,dx - \int e^{-i\langle x,\xi_0\rangle} f(x)\psi_\lambda(y-x)\,dx$$

$$\int |h(y)|dy \le \int f(x)\int |\psi_\lambda(y) - \psi_\lambda(y-x)|\,dy\,dx$$

Put $G_\lambda(x) = \int |\psi_\lambda(y) - \psi_\lambda(y-x)|\,dy$

$$= \int \left|\lambda^{-n}\psi\left(\tfrac{y}{\lambda}\right) - \lambda^{-n}\psi\left(\tfrac{y-x}{\lambda}\right)\right| dy$$

$$\le 2\int |\psi(y)|\,dy = 2\|\psi\|_{L^1}.$$

For fixed x,

$$\lim_{\lambda\to\infty} G_\lambda(x) = \lim_{\lambda\to\infty}\int \left|\lambda^{-n}\psi\left(\tfrac{y}{\lambda}\right) - \lambda^{-n}\psi\left(\tfrac{y-x}{\lambda}\right)\right| dy$$

$$= \lim_{\lambda\to\infty}\int \left|\psi(y) - \psi\left(y - \tfrac{x}{\lambda}\right)\right| dy$$

$$= 0, \quad \text{since } \psi \in \mathcal{S}.$$

Hence by DCT,

$$\lim_{\lambda\to\infty}\int |h(y)|\,dy \le \lim_{\lambda\to\infty}\int |f(x)|\,G_\lambda(x)\,dx = 0.$$

$\therefore\ \exists$ ~~λ₀~~ large such that

$$\|h\|_{L^1} < \epsilon$$

Put $V_0 = B\left(\frac{2}{\lambda}\right)$. Then the Lemma follows.

Lemma: Let $g \in L^\infty$ and $f \in L^1$ such that ~~$\hat{f}(\xi_0) \neq 0$ for some ξ_0~~.

$$g * f \equiv 0.$$

Then $\operatorname{Supp} \hat{g} \subset \{\xi ; \hat{f}(\xi) \neq 0\}$.

Proof: Let ξ_0 be such that $\hat{f}(\xi_0) \neq 0$. Let $f_1 = \frac{f(\xi)}{\hat{f}(\xi_0)}$.

Then

$$(g * f_1) = \frac{(g * f_0)}{\hat{f}(\xi_0)} \equiv 0.$$

and $\hat{f}_1(\xi_0) = 1$.

Let $\epsilon > 0$ choose V_0 such that from Lemma. $\exists h \in L$ with

$$\begin{cases} \hat{h}(\xi) = 1 - \hat{f}_1(\xi) & \forall \xi \in V_0 + \xi_0 = V_{\xi_0} \\ \|h\|_{L^1} < \epsilon < 1 \end{cases}$$

Let $\hat{\varphi} \in C_0^\infty(V_{\xi_0})$, Then

$$\hat{\varphi}(\xi)\hat{h}(\xi) = \hat{\varphi}(\xi) - \hat{\varphi}(\xi)\hat{f}_1(\xi)$$

or

$$\hat{\varphi}(\xi)(1 - \hat{h}(\xi)) = \hat{\varphi}(\xi)\hat{f}_1(\xi).$$

$$\hat{\varphi}(\xi) = \sum \hat{h}^n(\xi)\hat{\varphi}(\xi)\hat{f}_1(\xi)$$

$$\varphi(x) = (G * f_1)(x)$$

where

$$G = \sum h^n * \varphi \in L^1 \text{ since } (h^n = \underbrace{h * \cdots * h}_{n\text{-times}})$$

$$\|G\|_{L^1} \leq \sum \|h^n\|_{L^1} \|\varphi\|_{L^1}$$

$$\leq \left(\sum \|h\|_{L^1}^n\right)\|\varphi\|_{L^1} \leq \left(\frac{1}{1 - \|h\|_{L^1}}\right)\|\varphi\|_{L^1} < \infty.$$

Hence

$$\langle \hat{g}, \hat{\varphi} \rangle = \langle g, \check{\varphi} \rangle = \cancel{g*} (g * \varphi)(0)$$

$$= (g * G * f_1)(0)$$

$$= (G * (g * f_1))(0) \equiv 0 .$$

Hence $\hat{g} = 0$ in $V_{\xi_0} \Rightarrow \operatorname{supp} \hat{g} \subset \{\xi : \hat{f}(\xi) = 0\}$.

Tauberian Theorem of Weiner.

Let $W \subset L^1(\mathbb{R}^n)$ be a subspace such that

(1) $f \in W \Rightarrow$ ~~f~~ $f(x - x_0) \in W$ for all $x_0 \in \mathbb{R}^n$.

(2) $\displaystyle\bigcap_{f \in W} \{ \xi ; \hat{f}(\xi) = 0 \} = \phi$.

Then W is a dense subspace of $L^1(\mathbb{R}^n)$.

Proof: By Hahn-Banach Theo., $\exists\, g \in L^\infty(\mathbb{R}^n)$ such that

$\langle g, f \rangle = 0 \quad \forall f \in W$. i.e

$$\int g(x) f(x) dx = 0$$

Since $f \in W$, $f(x + x_0) \in W \ \forall x_0 \in \mathbb{R}^n \Rightarrow$

$$\int g(x) f(x + x_0) dx = 0$$

$$\Rightarrow \int g(-x) f(x_0 - x) dx = 0 .$$

Let ~~$g_1 = g$~~ $g_1 = \check{g}$, then

$$(g_1 * f)(x_0) = \int g_1(x) f(x_0 - x) dx = 0 .$$

Then

$$\operatorname{Supp} \hat{g}_1 \subset \{\xi : \hat{f}(\xi) = 0\} \quad \forall f \in W$$

$$\Rightarrow \operatorname{Supp} \hat{g}_1 \subset \bigcap_{f \in W} \{\xi : \hat{f}(\xi) = 0\} = \phi$$

$$\Rightarrow g_1 \equiv 0 \Rightarrow \overline{W} = L^1.$$

This proves the theorem.

Immediate corollary is

Corollary: Let $K \in L^1$ such that $\hat{K}(\xi) \neq 0 \ \forall \xi \in \mathbb{R}^n$. Let $W = \{$Space generated by translate of $K\}$. Then $\overline{W} = L^1(\mathbb{R}^n)$.

Tauberian theorem.

(1) Let $a \in \mathbb{R}^n$, $K \in L^1$ such that $\hat{K}(\xi) \neq 0 \ \forall \xi \in \mathbb{R}^n$. and let $\varphi \in L^\infty(\mathbb{R}^n)$ such that

$$\lim_{|x| \to \infty} (\varphi * K)(x) = a \hat{K}(0).$$

Then $$\lim_{|x| \to \infty} (\varphi * f)(x) = a \hat{f}(0) \quad \forall f \in L^1(\mathbb{R}^n).$$

(2) Assume that φ is uniformly continuous at ∞. i.e $\forall \epsilon > 0 \ \exists \lambda > 0$ ~~such that for $|x| > \lambda, |y| > \lambda$~~ $\delta > 0$ such that $\forall |x| > \lambda, |y| > \lambda$ $|x - y| < \delta$

$$|\varphi(x) - \varphi(y)| < \delta$$

Then $$a = \lim_{|x| \to \infty} \varphi(x).$$

Proof:

Let
$$W = \{ f \in L^1 ;\ \lim_{|x|\to\infty} \varphi * f(x) = a\hat{f}(0) \}.$$

Claim
Then W is a closed translation invariant subspace.

For $x_0 \in \mathbb{R}^n$, ~~$(f(x+x_0) * \varphi$~~

$$\big(\varphi * f(\cdot + x_0)\big)(x) = \int \varphi(y) f(x_0 + x - y)$$
$$= \big(\varphi * f\big)(x_0 + x).$$

Then
$$\lim_{|x|\to\infty} \big(\varphi * f(\cdot + x_0)\big) = \lim_{|x|\to\infty} (\varphi * f)(x_0 + x)$$
$$= \lim_{|x_0 + x|\to\infty} (\varphi * f)(x_0 + x)$$
$$= a\hat{f}(0).$$

Let $\{f_l\} \subset W$ be a Cauchy sequence. Let $f_l \to f$ in L^1

Then $\hat{f}_l(0) \to \hat{f}(0)$ as $l \to \infty$.

$$\| \varphi * f_l - \varphi * f \|_{\infty} \le \|\varphi\|_{L^\infty} \|f_l - f\|_{L^1} \to 0$$

$$\Rightarrow \lim_{|x|\to\infty} (\varphi * f)(x) = \lim_{|x|\to\infty} (\varphi * f_l)(x) = \lim_{l\to\infty} a\hat{f}_l(0) = a\hat{f}(0).$$

~~Hence by Weiner then,~~ ~~$(\varphi * \ldots) * f =$~~

By the hypoth. $K \in W$. Hence by Weiner thm, $W = L^1$ and this prove (1).

(2) Let $\epsilon > 0$, $\delta > 0$, $\lambda > 0$ s.t.
$$|\varphi(x) - \varphi(y)| < \epsilon \quad \forall\ |x-y| < \delta,\ |x| > \lambda,\ |y| > \lambda.$$

Let $0 \le f \in C_0^\infty(\mathbb{R}^n)$ s.t. $f(x) = 0$ for $|x| > \delta$, $\hat{f}(0) = 1$

Then by 1

$$\lim_{|x|\to\infty} (f*\varphi)(x) = a.$$

$$\varphi(x) - (f*\varphi)(x) = \int_{|y|<\delta} \left(\varphi(x) - \varphi(x-y)\right) f(y)\,dy$$

~~$\lim_{|x|\to\infty} \varphi(x) - \lim_{|x|\to\infty}(f*\varphi)(x) =$~~ Hence for $|x| > \lambda$, $|x-y| > \lambda$,

$$|\varphi(x) - (f*\varphi)(x)| < \varepsilon \int |f(y)|\,dy = \varepsilon$$

Hence $\lim_{|x|\to\infty} |\varphi(x) - a| < \varepsilon \to \lim_{|x|\to\infty} \varphi(x) = a.$

This proves the theorem.