Linear Evolution Equations: Linear Parabolic PDE

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1 Introduction

In these lectures, we discuss the existence and uniqueness of weak solution to the following class of second order linear parabolic differential equations:

$$\frac{\partial u}{\partial t} + \mathcal{A}(t)u = f \tag{1.1}$$

and initial condition:

$$u(0) = u_0,$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and $T > 0$ fixed. Further $f : \Omega \times (0, T] \to \mathbb{R}$ and $u_0 : \Omega \to \mathbb{R}$ are given functions in their respective domains of definition. Here $u = u(x, t)$ defined on $\Omega \times (0, T]$ is unknown and $\mathcal{A}(t)$ is an elliptic differential operator. Note that we, at this stage, do not spell out the form of $\mathcal{A}$ and also the boundary condition.

Since we are dealing with functions in space-time domain, therefore, in the beginning, we discuss Banach space valued distributions and function spaces. Then, we define weak formulation and establish abstract theory for solvability of the weak formulation.

2 Banach Space Valued Distributions and Function Spaces

Since the problem (1.1) is defined on a space-time domain, we shall study some function spaces defined on space-time domain.

Let X be a Banach space with $\|\cdot\|_X$. We now denote X-valued L^p spaces by $L^p(0, T; X)$, which consists of all strongly measurable functions $v : (0, T] \to X$ such that

$$\|v\|_{L^p(0,T;X)} := \Big(\int_0^T \|v(t)\|_X^p dt \Big)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty$$

and for $p = \infty$

$$\|v\|_{L^p(0,T;X)} := \text{essup}_{0 \leq t \leq T} \|v(t)\|_X < \infty.$$

The space $C([0, T]; X)$ consists of all continuous $v : [0, T] \to X$ such that

$$\|v\|_{C([0,T];X)} := \max_{0 \leq t \leq T} \|v(t)\|_X < \infty.$$

Definition 2.1 *Weak Derivative: For $u \in L^1(0, T; X)$, $v \in L^1(0, T; X)$ is called its weak derivative, that is, $u_t = v$ if*

$$\int_0^T \phi_t(t)u(t)dt = -\int_0^T \phi(t)v(t)dt$$

for all scalar test functions $\phi \in \mathcal{D}(0, T)$. Here, $\mathcal{D}(0, T)$ is test space defined on $(0, T)$, that is, it is the space of infinitely differentiable functions with compact support in $(0, T)$.

Definition 2.2 *(Space-time Sobolev Space): The space $W^{1,p}(0, T; X)$ is defined as*

$$W^{1,p}(0, T; X) := \Big\{ u \in L^p(0, T; X) : u_t \text{ exits and } u_t \in L^p(0, T; X) \Big\}.$$

On $W^{1,p}(0,T;X)$, define its norm as

$$\|u\|_{W^{1,p}(0,T;X)} := \Big(\int_0^T (\|u(t)\|_X^p + \|u_t(t)\|_X^p)dt\Big)^{\frac{1}{p}}, \quad 1 \le p < \infty$$

and for $p = \infty$

$$\|v\|_{W^{1,\infty}(0,T;X)} := \text{essup}_{0\le t\le T}(\|u(t)\|_X + \|u_t(t)\|_X).$$

Hence forward for $p = 2$, we write

$$H^1(0,T;X) := W^{1,2}(0,T;X)$$

Below, we state without proof two theorems on calculus in an abstract space. For a proof, we refer to pp. 286-288 of Evans [1]

Theorem 2.1 *Let $u \in W^{1,p}(0,T;X),\ 1 \le p \le \infty$. Then the followings hold:*

(i) $u \in C([0,T];X)$ after eventual modification on a set of measure zero.

(ii) $u(t) = u(s) + \int_s^t u^{'}(\tau)\, d\tau \quad 0 \le s \le t \le T$.

(iii) Further,

$$\max_{t\in[0,T]} \|u(t)\|_X \le C\ \|u\|_{W^{1,p}(0,T;X)}$$

where C depends on T.

Note in Theorem 2.1, u and $u^{'} \in L^p(0,T;X)$. Now what can be said if u and $u^{'}$ belong to different spaces and the answer to this question can be founf from the results of the following Theorem.

Theorem 2.2 *Let $u \in L^p(0,T;V)$ and $u_t \in L^p(0,T;V^{'})$ where $V^{'}$ is the dual space of V with $V \hookrightarrow H = H^{'} \subset V^{'}$. Then, followings hold:*

(i) $u \in C([0,T];H)$ after possible modification on a set of measure zero.

(ii) The mapping $t \to \|u(t)\|_H$ is absolutely continuous with

$$\frac{d}{dt}\|u(t)\|_H^2 = 2\langle u_t(t), u(t)\rangle \text{ for a.e. } t \in [0,T].$$

(iii) Moreover, there is a positive constant $C = C(T)$ such that

$$\max_{t\in[0,T]} \|u(t)\|_H \le C\ (\|u(t)\|_{L^2(0,T;V)} + \|u_t(t)\|_{L^2(0,T;V')}).$$

3 Abstract Formulation and Wellposedness

Given two separable Hilbert spaces H and V with dual $H^{'}$ of H identified as H, consider the Gelfand triplet

$$V \hookrightarrow H = H^{'} \hookrightarrow V^{'} \tag{3.2}$$

where $\hookrightarrow$ is continuous and dense embedding and $V^{'}$ is the dual of V. We now denote by $(\cdot,\cdot)$ an inner product in H and $\langle\cdot,\cdot\rangle$ duality parring between $V^{'}$ and V. Note that the following relation holds for $v \in H$ and $w \in V^{'}$

$$\langle v, w\rangle = (v, w).$$

Below, we make the following assumptions:

(A1) $\mathcal{A}(t) \in \mathcal{L}(V, V^{'})$ depends continuously on $t \in [0, T]$

Now associate with $\mathcal{A}(t)$, a bilinear form on V given by

$$v, w \mapsto a(t; v, w) \text{ for each } t \in [0, T].$$

which satisfies

$$a(t; v, w) = \langle \mathcal{A}(t)v, w\rangle. \tag{3.3}$$

Further assume that the bilinear form satisfies the following Garding type inequality:

(A2) For $v \in V$ there exist real constants $\alpha > 0$ and β such that

$$\langle \mathcal{A}(t)v, w\rangle = a(t; v, w) \geq \alpha\|v\|_V^2 - \beta\|w\|_H^2.$$

Now consider the following abstract evolution problem: For a given $f \in L^2(0, T; V^{'})$ and $u_0 \in H$ find $u \in L^2(0, T; V)$ with $u_t \in L^2(0, T; V^{'})$ satisfying

$$\frac{du}{dt} + \mathcal{A}(t)u = f(t) \quad \text{in} \;\; V', \;\; \text{for a.e } t \in [0, T], \tag{3.4}$$

with initial condition

$$u(0) = u_0. \tag{3.5}$$

Below, we establish the main theorem on solvability of the problem (3.4)-(3.5).

Theorem 3.1 *Let H, V and $\mathcal{A}(t)$ be as given above. Further, let assumptions (A1)-(A2) hold. Then for a given $f \in L^2(0, T; V^{'})$ and $u_0 \in H$, the problem (3.4)-(3.5) has a unique solution $u \in L^2(0, T; V)$ with $u_t \in L^2(0, T; V^{'})$.*

Proof: We shall first prove uniqueness. Assume that the solution is not unique, that is, u_1 and u_2 are two distinct solutions of (3.4)-(3.5) with $u_1 \neq u_2$. Note, u_i satisfies

$$\frac{du_i}{dt} + \mathcal{A}(t)u_i = f, \tag{3.6}$$

$$u_i(0) = u_0. \tag{3.7}$$

With $w = u_1 - u_2$, now u satisfies

$$\frac{dw}{dt} + \mathcal{A}(t)w = 0, \tag{3.8}$$

with

$$w(0) = 0. \tag{3.9}$$

Taking duality between w and (3.8), we arrive at

$$\langle \frac{dw}{dt}, w\rangle + \langle \mathcal{A}(t)w, w\rangle = 0$$

Using (3.3) and (ii) of Theorem 2.2, we obtain

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_H^2 + a(t; w, w) = 0. \tag{3.10}$$

Applying Garding type inequality for the bilinear form $a(t; w, w)$ and find that (3.10) becomes

$$\frac{d}{dt}\|w(t)\|_H^2 + 2\alpha\|w(t)\|_V^2 - 2\beta\|w(t)\|_H^2 \leq 0 \tag{3.11}$$

Using integrating factor $e^{-2\beta t}$, we rewrite (3.11) as

$$\frac{d}{dt}(e^{-2\beta t}\|w(t)\|_H^2) + 2\alpha e^{-2\beta t}\|w(t)\|_V^2 \leq 0 \tag{3.12}$$

and hence, integrating with respect to t from 0 to t^*, we obtain

$$e^{-2\beta t^*}\|w(t^*)\|_H^2 + 2\alpha\int_0^{t^*} e^{-2\beta s}\|w(s)\|_V^2 ds \leq 0.$$

Therefore, $w = 0$, that is, $u_1 = u_2$ and it leads to a contradiction. Hence, the solution of (3.4)-(3.5) is unique.

For existence, we use Bubnov-Galerkin method. Assume that $\{\phi\}_{j=1}^{\infty}$ forms a basis of V in the sense that for every m; $\{\phi_1, \phi_2, \cdots, \phi_m\}$ are linearly independent and the linear combinations $\sum_{j=1}^{m} \xi_j\phi_j,\ \xi_j \in \mathbb{R}$ are dense in V.

For a fixed m, let $V_m = \text{span}\{\phi_1, \phi_2, \cdots, \phi_m\}$ and let P_m be the orthogonal projection from H onto V_m. We now seek a function $u_m : [0, T] \to V_m$ of the form

$$u_m(t) := \sum_{j=1}^{m} g_{jm}(t)\phi_j, \tag{3.13}$$

where g_{jm}'s are chosen so that

$$(\frac{d}{dt}u_m(t), \phi_k) + a(t; u_m(t), \phi_k) = \langle f(t), \phi_k\rangle, \quad 1 \leq k \leq m \tag{3.14}$$

and

$$u_m(0) = P_m u_0 := \sum_{j=1}^{m} \xi_{jm}\phi_j. \tag{3.15}$$

with

$$P_m u_0 := \sum_{j=1}^{m} \xi_{jm}\phi_j \to u_0 \ \text{ in } H \text{ as } m \to \infty \tag{3.16}$$

The system (3.14)-(3.15) leads to a system of linear ODE and hence, by Picard's theorem there exists a unique solution to (3.14)-(3.15). Now, it remains to show that $\lim_{m\to\infty} u_m(t) = u(t)$ and the limiting function u is a solution of (3.4)-(3.5).

Multiply (3.14) by $g_{km}(t)$ and summing over k, we arrive at

$$(\frac{d}{dt}u_m(t), u_m(t)) + a(t; u_m(t), u_m(t)) = (f(t), u_m(t))$$

and hence,

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|_H^2 + a(t; u_m(t), u_m(t)) = \rangle f(t), u_m(t)\rangle. \tag{3.17}$$

For $\langle f(t), u_m(t)\rangle$, use Cauchy-Schwartz to arrive at

$$\langle f(t), u_m(t)\rangle \le \|f(t)\|_{V'}\|u_m(t)\|_V \tag{3.18}$$

Use Young's inequality $ab \le \frac{1}{2\epsilon}a^2 + \frac{\epsilon}{2}b^2 \quad a, b \ge 0, \epsilon > 0$ to (3.18) to find that

$$\langle f(t), u_m(t)\rangle \le \frac{1}{2\epsilon}\|f(t)\|_{V'}^2 + \frac{\epsilon}{2}\|u_m(t)\|_V^2 \tag{3.19}$$

On substituting (3.19) in (3.17) and for the bilinear form $a(t;\cdot,\cdot)$, use Garding type inequality with $\epsilon = \alpha$, we obtain

$$\frac{d}{dt}\|u_m(t)\|_H^2 + \alpha\|u_m(t)\|_H^2 \le \frac{1}{\alpha}\|f(t)\|_{V'}^2 + 2\beta\|u_m(t)\|_H^2. \tag{3.20}$$

Setting $y(t) = \|u_m(t)\|_H^2$ and $z(t) = \|f(t)\|_{V'}^2$, we rewrite (3.20) as

$$\frac{d}{dt}y(t) \le \frac{1}{\alpha}z(t) + 2\beta y(t). \tag{3.21}$$

Apply Gronwall's inequality to obtain

$$y(t) \le e^{2\beta t}(y(0) + \frac{1}{\alpha}\int_0^t z(s)\, ds). \tag{3.22}$$

Note that

$$y(0) = \|u_m(0)\|_H^2 \le C\|u_0\|^2.$$

Thus, we arrive at

$$\|u_m(t)\|_H^2 \le C(\|u_0\|^2 + \int_0^t \|f(s)\|_{V'}^2\, ds).$$

Taking maximum in time from 0 to T, we find that

$$\max_{0\le t\le T}\|u_m(t)\|_H^2 \le C(\|u_0\|^2 + \int_0^T \|f(s)\|_{V'}^2\, ds),$$

and hence,

$$\|u_m\|_{L^\infty(0,T;H)}^2 \le C(\|u_0\|^2 + \|f\|_{L^2(0,T;V')}^2). \tag{3.23}$$

Again integrate (3.20) with respect to t from $(0,T]$ to obtain

$$\|u_m\|^2_{L^2(0,T;V)} := \int_0^T \|u_m(s)\|^2_V ds \le C(T,\alpha)(\|u_0\|^2 + \|f\|^2_{L^2(0,T;V')}). \tag{3.24}$$

As a consequence, the sequence $\{u_m\}$ is bounded uniformly in the Hilbert space $L^2(0,T;V)$. By weak compactness, we can extract a subsequence called $\{u_{m_l}\} \subset L^2(0,T;V)$ such that

$$u_{m_l} \rightharpoonup u \;\text{ weakly in } L^2(0,T;V). \tag{3.25}$$

Let N be fixed, but arbitrary with $m_l > N$. Note that (3.17) is valid with replacing m by m_l. Then multiply the resulting equation by $\psi(t)$ where

$$\psi(t) \in C^1[0,T] \;\text{ with } \psi(T) = 0, \tag{3.26}$$

and integrate over $(0,T]$. Then, choose $\psi_N = \psi\phi_N$ to obtain

$$\int_0^T \{-(u_{m_l}(t), \psi_N^{'}(t)) + a(t; u_{m_l}(t), \psi_N(t))\}dt = \int_0^T \langle f(t), \psi_N(t)\rangle\, dt + (u_{0m_l}, \psi_N(0)). \tag{3.27}$$

By (3.25), we now pass the limit in (3.27) as $m_l \to \infty$ to find that

$$\int_0^T \{-(u(t), \psi_N^{'}(t)) + a(t; u(t), \psi_N(t))\}\, dt = \int_0^T \langle f(t), \psi_N(t)\rangle\, dt + (u_0, \psi_N(0)). \tag{3.28}$$

Note that (3.28) holds for any ψ satisfying (3.26). Hence, the equation (3.28) makes sense if $\psi \in \mathcal{D}(0,T)$. With $\psi \in \mathcal{D}(0,T)$, (3.28) reduce to

$$\langle \frac{du}{dt}(t), \phi_N\rangle + a(t; u(t), \phi_N) = \langle f(t), \phi_N\rangle. \tag{3.29}$$

Here, the derivative is taken in the sense of distribution, that is, in $\mathcal{D}^{'}(0,T)$. Thus,

$$\langle \frac{du}{dt}(t), \phi_N\rangle + \langle \mathcal{A}(t)u(t), \phi_N\rangle = \langle f(t), \phi_N\rangle. \tag{3.30}$$

Note that in (3.29), N can be arbitrary.

Since finite linear combinations of $\{\phi_j\}$ are dense in V, the equation (3.29) is valid for any $v \in V$ and hence, we arrive at

$$\frac{du}{dt} = -\mathcal{A}(t)u + f \;\text{ in }\; V^{'}, \tag{3.31}$$

as $\dfrac{du}{dt} = -\mathcal{A}(t)u + f \in L^2(0,T;V^{'})$. Thus, $u \in L^2(0,T;V)$ and $u_t \in L^2(0,T;V^{'})$.

In oder to obtain $u(0) = u_0$, note that from (3.27) holds true if ψ_N is replaced by $\psi \in C^1([0,T];V)$ and obtain

$$\int_0^T \{-(u_{m_l}(t), \psi_t(t)) + a(t; u_{m_l}(t), \psi(t))\}\, dt = \int_0^T (f(t), \psi(t))\, dt + (u_{m_l}(0), \psi(0)). \tag{3.32}$$

Taking limit as $m_l \to \infty$ we arrive at

$$\int_0^T \{-(u(t), \psi_t(t)) + a(t; u(t), \psi(t))\}\, dt = \int_0^T (f(t), \psi(t))\, dt + (u_0, \psi(0)). \tag{3.33}$$

On the other hand multiply (3.31) by ψ and integrate to obtain

$$\int_0^T \{-(u(t), \psi_t(t)) + a(t; u(t), \psi(t))\}\, dt = \int_0^T (f(t), \psi(t))\, dt + (u(0), \psi(0)). \tag{3.34}$$

Compare (3.33) with (3.34) to arrive at

$$(u_0, \psi(0)) = (u(0), \psi(0)).$$

Since ψ is arbitrary, we now obtain

$$u(0) = u_0,$$

and this completes the rest of the proof.

3.1 Applications

Consider the following linear parabolic initial and boundary value problem: Find $u(x,t)$ in $\Omega \times (0, \infty)$ such that

$$\frac{\partial u}{\partial t} + \mathcal{L}(t) u = f, \quad x \in Q_T := \Omega \times (0, T], \tag{3.35}$$

$$u(x,t) = 0, \quad x \in \partial Q_T := \partial\Omega \times (0, T], \tag{3.36}$$

$$u(x,0) = u_0, \quad x \in \Omega, \tag{3.37}$$

where Ω is a bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial\Omega$, and

$$\mathcal{L}\phi := -\sum_{j,k=1}^{d} \frac{\partial}{\partial x_k}\Big(a_{jk} \frac{\partial \phi}{\partial x_j}\Big) + \sum_{j=1}^{d} b_j \frac{\partial \phi}{\partial x_j} + a_0 \phi.$$

Below, we make reasonable assumptions on the coefficients, on f and u_0.

Assumptions. Assume that

(i) the elliptic operator $\mathcal{L}$ is elliptic in the sense that there is a positive constant $\alpha_0 > 0$ such that

$$\sum_{j,k=1}^{d} a_{jk}\, \xi_j \xi_k \geq \alpha_0 \sum_{j=1}^{d} |\xi_j|^2 \quad \forall 0 \neq \xi \in \mathbb{R}^d.$$

(ii) $a_{jk}; b_j, a_0 \in L^\infty(Q_T)$ with $a_{jk} = a_{kj}$.

(iii) $f \in L^2(Q_T)$ and $u_0 \in L^2(\Omega)$.

We now associate with the elliptic operator $\mathcal{L}$ a bilinear form $a(t, \cdot, \cdot)$ as

$$a(t; v, w) := \sum_{j,k=1}^{d} \int_\Omega a_{jk} \frac{\partial v}{\partial x_j} \frac{\partial w}{\partial x_k}\, dx + \sum_{j=1}^{d} \int_\Omega \frac{\partial v}{\partial x_j} w\, dx + \int_\Omega v\, w\, dx, \quad v, w \in H_0^1(\Omega), \quad \text{a.e. } t \in (0, T].$$

To put our problem in the abstract framework like (3.4)-(3.5), we now associate with $u(x,t)$ a mapping $u : (0,T] \longrightarrow H_0^1(\Omega)$ defined by

$$[u(t)](x) = u(x,t), \quad x \in \Omega, \ t \in [0,T].$$

Essentially for fixed t in $(0,T]$, $u(t) \in H_0^1(\Omega)$. Similarly, $f(x,t)$ can be defined as a map $f : [0,T] \longrightarrow L^2(\Omega)$ given by

$$[f(t)](x) = f(x,t), \quad x \in \Omega, \ t \in [0,T].$$

Moreover, because of our assumption it is easy to check that the bilinear form satisfies the following estimates:

- **Boundedness.** There is a positive constant M such that the biliear form is bounded in the sense that for $v, w \in H_0^1(\Omega)$,

$$|a(t,v,w)| \leq M\|v\|_{H_0^1(\Omega)} \, \|w\|_{H_0^1(\Omega)}.$$

- **Garding type inequality.** There exist two real constant $\alpha > 0$ and β such that for $v \in H_0^1(\Omega)$,

$$a(t;v,v) \geq \alpha\|v\|^2_{H_0^1(\Omega)} - \beta\|v\|_{L^2(\Omega)}.$$

Problem 3.1 *Verify above two properties for the bilinear form.*

Since for fixed $t \in (0,T]$ and fixed $v \in H_0^1(\Omega)$, the bilinear form $a(t;v,\cdot)$ can be thought of as a linear form on $H_0^1(\Omega)$, which is bounded because of the boundedness of the biliear form, therefore, by Ritz-Representation theorem, we can associate with this an abstract bounded linear operator $\mathcal{A}(t) : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ such that

$$\langle \mathcal{A}(t)v, w\rangle := a(t;v,w), \ \ w \in H_0^1(\Omega), \tag{3.38}$$

where $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$, and $\langle\cdot,\cdot\rangle$ is the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Set $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$. Note that $V' = H^{-1}(\Omega)$ is the dual space of V and it is easy to check that V, H, V' forms a Gelfand triplet. Thus, one can write (3.35) -(3.37) in abstract form as:

$$u_t + \mathcal{A}(t)u = f(t) \quad \text{in } V'$$

with $u(0) = u_0 \in H$.

Note the using boundedness of the bilinear form, the hypothesis $(A1)$ is satisfied and further, due to Garding type inequality, the operator $\mathcal{A}(t)$ satisfies $(A2)$. Therefore, we apply Theorem 3.1 to discuss existence of a unique weak solution of (3.35)-(3.37).

References

[1] L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, Vol. 19, AMS, Providence, Rhode Island, 1998 (Reprinted 2002).