# Introduction to Exact Controllability and Observability; Variational Approach and Hilbert Uniqueness Method [1]

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We plan to discuss the following topics in these lectures

1. A brief introduction

2. Review of finite dimensional systems: controllability, observability etc.

3. Exact controllability of linear wave equation: Variational approach, observability inequality.

4. Observability inequality in 1D via Ingham's inequality

5. Hilbert Uniqueness Method: Motivation, multiplier method, generalization.

## 1. **Introduction**

Any control problem will consists of the following:

(i) a set of equations known as *state equations* which we call a controlled system; this is an input-output system. State equations involve (a) input function, called controls and (b) output known as the state of the system, corresponding to the given input (control).

(ii) an observation of the output of the controlled system (partial information).

(iii) an objective to be achieved.

The set of equations can appear in different forms like; ODE (finite dimensional control systems), PDE (infinite dimensional set-up), integral equations and so on. PDE's can be of different types; elliptic, Parabolic or hyperbolic. The controls can appear in a distributed way, through the boundary or through a certain part of the domain, boundary etc.

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**Various objectives**: (i) Minimize certain criteria depending on the state and/or observations, control etc. For example minimizing the energy/cost, time or maximizing the profit etc.(optimal control problem)

(ii) Look for controls so that the state belongs to a certain target set (controllability problem).

(iii) Look for controls which stabilizes the state or observations (stabilization problem).

**Finite Dimensional Model (Linear)**: This can be described by a system of ODEs of the form $\frac{dy}{dt} = Ay + Bu, y(t_0) = y_0$, where A is an $n \times n$ matrix and B is an $m \times n$, matrix, where $m, n \in \mathbb{N}$ and $m \leq n$. Normally $m < n$ which indicates that the number of control variables are smaller than the number of states to be controlled. A control problem can be stated as follows; Given a target $y_1 \in \mathbb{R}^n$, find a control $u$ and a time $T > 0$ so that the corresponding solution $y = y(t)$ satisfies $y(T) = y_1$.

We will quickly review, at a later stage, some aspects of this issue which was already introduced through other lectures.

**Remark**: The modelling in terms of finite and infinite dimensional (PDE) systems is very important in practice as it has quite different properties from a control theoretic point of view. In fact, even the analysis varies according to the class of PDE's, for example, the nature of PDE's, say, whether it is parabolic or hyperbolic and its different characteristic properties play an important role in the controllability results. In hyperbolic equations, we have the notion of *finite speed of propagation* and *evolution of singularities* (non- smooth) where as the heat equation posses *infinite speed of propagation* and *smoothing effect*. The notion of *Exact controllability* is a suitable notion in hyperbolic problems but *smoothing effect* in parabolic problems force us to look for *approximate controllability* results.

Again due to finite speed of propagation any given data (control) takes certain amount of time to reach other parts of the domain and hence controllability (exact) could be achieved only at a sufficiently large time. This is not the case in heat equations. In elliptic problems (no time as it is equilibrium case), one look for optimal control problems. Of course optimal control problem are also relevant in parabolic and hyperbolic equations.

Indeed, it is not possible to discuss various issues as mentioned earlier, in this short course. We mainly, restrict ourselves to the case of wave equation (hyperbolic). We present the *variational approach* and introduce *Hilbert Uniqueness Method (HUM)*.

**Examples: (1) Elliptic equation**: For an electric potential $\phi$ in a domain $\Omega$ occupied by the electrolyte, $\phi$ satisfies

$$\begin{cases} -div(a\nabla\phi) = 0 \ in \ \Omega, \\ -\sigma\frac{\partial\phi}{\partial\nu} = i \ on \ \Gamma_a, -\sigma\frac{\partial\phi}{\partial\nu} = 0 \ on \ \Gamma_r - \sigma\frac{\partial\phi}{\partial\nu} = f(\phi) \ on \ \Gamma_c, \end{cases} \tag{1.1}$$

where $\partial\Omega = \Gamma = \Gamma_a \cup \Gamma_r \cup \Gamma_c$, $\Gamma_a$ is anode, $\Gamma_c$ i s cathode and $\Gamma_r$ is the rest of the boundary. The control function is the current density $i$, $\sigma$ is the conductivity and $f$ is known as cathode-polarization function. The problem is to minimize

$$(P_1) \qquad Inf\left\{ J_1(\phi) : (\phi, i) \in H^1(\Omega) \times L^2(\Gamma_a) \ where \ (\phi, i) \ satisfies \ (1.1) \right\}$$

where

$$J_1(\phi) = \int_{\Gamma_c} (\phi - \bar{\phi})^2$$

The cathode is protected if the electric potential is close to a given potential $\bar{\phi}$ on $\Gamma_c$. One has to choose the current $i$ so that $J_1$ is minimized.

A compromise between the *cathodic protection* and *consumed energy* can be obtained by looking at the problem

$$Inf \ J_2(\phi),$$

where $J_2(\phi) = \int_\Omega (\phi - \bar{\phi})^2 + \beta \int_{\Gamma_a} i^2$ for all $(\phi, i) \in H^1 \times L^2(\Gamma_a)$.

**(2) Parabolic Equation (Identification of a source of pollution):** The concentration of a pollutant $y(x,t)$ satisfies the parabolic PDE

$$\begin{cases} \frac{\partial y}{\partial t} - \Delta y + V.\nabla y + \sigma y = s(t)\delta_a \ in \ \Omega \times (0,T) \\ \frac{\partial y}{\partial \nu} = 0 \ on \ \Gamma \times (0,T), \ \ y(x,0) = y_0 \ in \ \Omega \end{cases} .$$

Here $a \ \epsilon \ K$ is the position of source of pollution in a compact set $K \subset \bar{\Omega}$ and $s(t)$ is the flow rate of pollution.

Assume that the pollutant $y$ can be observed in a region $O \subset \Omega$, denoted by $y_{obs}$ in an interval of time $[0,T]$. The problem is to find $a \in K$ so that it minimizes $\int_0^T \int_O (y - y_{obs})^2$.

One can also have other problems, where the source is known, but not accessible and hence $s(t)$ is unknown. Hence find $s$ satisfying some appropriate bounds $s_o \leq s(t) \leq s$, and minimize the same functional as above.

In these lectures, we discuss the issue of *exact controllability* which can be formulated as follows. Given an evolution system (described by ODE/PDE), we are allowed to act on the trajectories (solutions) by means of a suitable control (either in a distributed way, that is acting through the equation in the full or partial domain or through the boundary ). Then, given a time interval [0,T] and initial and final states, the problem is to find a control such that corresponding solution matches both the given initial state at time $t = 0$ and final state at time $t = T$.

The research in this area in the last few decades is very intensive. We plan to sketch few things in the context of wave equation. We begin with the review of ODE (finite dimensional case).

## 2. **Controllability of finite dimensional systems**

Recall the controlled system described by the ODE system

$$\begin{cases} x'(t) = Ax(t) + Bu(t), \; t \in (0,T), \\ x(0) = x^0 \end{cases} . \tag{2.1}$$

Here A is $n \times n$ real matrix, B is an $n \times m$ real matrix, $x : [0,T] \to \mathbb{R}^n$ is the state and $u : [0,T] \to \mathbb{R}^m$ is the control function. Clearly $m \leq n$ and certainly we wish to use number of controls as minimum as possible, that is $m < n$.

The solution of (2.1) is given by the variational formula

$$x(t) = e^{At}x^0 + \int_0^t e^{A(t-s)}Bu(s)ds \tag{2.2}$$

**Definition(Controllability):** We say (2.1) is *controllable* in time $T > 0$, if for given $x^0, x^1 \in \mathbb{R}^n, \exists u \in L^2(0,T;\mathbb{R}^m)$ such that $x(t)$ satisfying (2.1) also satisfies $x(T) = x^1$. It is *Null controllable*, if $x(t)$ satisfies $x(T) = 0$.

**Proposition:** 2.1: For the finite dimensional linear systems, null controllability is equivalent to controllability. To see this first solve, $y' = Ay$ with $y(T) = x^1$, then solve for the null controllability of

$$z' = Az + Bu, z(0) = x^0 - y(0), z(T) = 0$$

. Then $x = y + z$ satisfies $x' = Ax + Bu, x(0) = x^0, x(T) = y(T) = x^1$.

**Remark:** Even for finite dimensional systems controllability is not always achieved.

**Example:** Consider the system $x_1' = x_1 + u$, $x_2' = x_2$. That is

$$x' = Ax + Bu \; where \; A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Clearly, $u$ does not influence the trajectory $x_2(t) = x_2^0 e^t$ and hence it is not controllable.

**Remark:** This doesn't mean that in a $2 \times 2$ system, one always needs two controls. There are $2 \times 2$ systems where one control will suffice to achieve the controllability.

**Example:** Consider the system $x_1' = x_2$, $x_2' = u - x_1$, that is

$$x' = Ax + Bu, A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Equivalently, $x_1'' + x_1 = u$ (harmonic oscillator). Unlike the previous equation, $u$ acts on the second equation, where both $x_1$ and $x_2$ are present. Hence one cannot immediately conclude the exact controllability or otherwise.

But the present system is in fact controllable. To see this, choose any function $z$ satisfying the initial and final conditions, namely $z(0) = x_1^0, z'(0) = x_2^0, z(T) = x_1^1, z'(T) = x_2^1$. Plenty of such functions exist. Now $x_1 = z, x_2 = z'$ with the control $u = z'' + z$ will solve the control problem.

**Equivalent criteria via observability system:**

Consider the adjoint system

$$\begin{cases} -\phi' = A^*\phi, t \in (0,T) \\ \phi(T) = \phi_T \end{cases} \tag{2.3}$$

where $A^*$ is the adjoint that satisfies $\langle Ax, y\rangle = \langle x, A^*y\rangle, \forall x, y \in \mathbb{R}^n$ and $\phi_T \in \mathbb{R}^n$. Multiplying (2.1) by $\phi$ and (2.3) by $x$, we get

$$\begin{aligned} \langle x', \phi\rangle &= \langle Ax, \phi\rangle + \langle Bu, \phi\rangle \\ &= \langle x, A^*\phi\rangle + \langle Bu, \phi\rangle \\ &= -\langle x, \phi'\rangle + \langle Bu, \phi\rangle \end{aligned}$$

which gives $\frac{d}{dt}\langle x, \phi\rangle = \langle Bu, \phi\rangle$. Integrating w.r.t. $t$, we get

$$\langle x(T), \phi_T\rangle - \langle x^0, \phi(0)\rangle = \int_0^T \langle u, B^*\phi\rangle.$$

Hence, we have the following proposition

**Proposition 2.2:** System (2.1) is null-controllable, that is $x(T) = 0$ if and only if

$$\int_0^T \langle u, B^*\phi\rangle + \langle x^0, \phi(0)\rangle = 0, \ \forall \phi_T \in \mathbb{R}^n. \tag{2.4}$$

for all $\phi_T \in \mathbb{R}^n$ and $\phi$ is the solution to (2.3).

We remark that for all $\phi_T$ , $x^0 \in \mathbb{R}^n$ the equation (2.4) is the optimality condition for the critical points of the quadratic functional $J : \mathbb{R}^n \to \mathbb{R}$ defined by

$$J(\phi_T) = \frac{1}{2}\int_0^T |B^*\phi|^2 + \langle x^0, \phi(0)\rangle,$$

where $\phi$ is the solution corresponding to (2.3). Suppose $\hat{\phi}_T$ is a minimizer of $J$, that is

$$J(\hat{\phi}_T) = Min\ J(\phi_T),$$

then using the fact that $\lim_{h\to 0} \dfrac{J(\hat{\phi}_T + h\phi_T) - J(\hat{\phi}_T)}{h} = 0$ for all $\phi_T \in \mathbb{R}^n$ (first principle), we see that

$$\int_0^T \langle B^*\hat{\phi}, B^*\phi\rangle dt + \langle x^0, \phi(0)\rangle = 0.$$

Then (2.4) implies that $u = B^*\hat{\phi}$ is a control driving the system $x^0$ to 0. Thus, we have

**Proposition 2.3:** Suppose $J$ has a minimizer $\hat{\phi}_T \in \mathbb{R}^n$ and $\hat{\phi}$ be the corresponding solution of the adjoint system (2.3) with data $\hat{\phi}(T) = \hat{\phi}_T$. Then $u = B^*\hat{\phi}$ is a control of system (2.1) with initial data $x^0$.

**Remark:** This is the variational method of obtaining a control if $J$ has a minimum. We also remark that by varying $J$, it may be possible to obtain different types of controls.

**Definition (Observability):** The system (2.3) is said to be observable in time $T > 0$ if $\exists c > 0$ such that

$(2.5)_a$ $\quad \int_0^T |B^*\phi|^2 \geq c|\phi(0)|^2$ (*observability inequality*)

for $\phi_T \in \mathbb{R}^n$ and $\phi$ is the solution of (2.3). This is equivalent to

$(2.5)_b$ $\quad \int_0^T |B^*\phi|^2 \geq c|\phi_T|^2.$

The equivalence follows from the fact that the map which associates $\phi_T \in \mathbb{R}^n$ to the vector $\phi(0) \in \mathbb{R}^n$ is a bounded linear operator with bounded inverse.

**Remark:** It basically tells us that, if we begins from $\phi(T) = \phi_T$ which evolves (reversely) according to the adjoint equation and observe the quantity $B^*\phi(t)$ for all $0 < t < T$, then $\phi(0)$ is uniquely determined. The above inequality is equivalent to the *unique continuation principle(u.c.p)*

(2.6) $\quad B^*\phi(t) = 0\ \forall\ t \in [0, T] \Rightarrow \phi_T = 0.$

Clearly, $(2.5)_b$ implies u.c.p. Conversely, if (2.6) is true, then $|\phi_T|_* = \left(\int_0^T |B^*\phi|^2\right)^{1/2}$ is a norm equivalent to the norm $|\phi_T|$ in $\mathbb{R}^n$ (finite dimensional), we have $(2.5)_b$.

**Remark:** In general, in infinite dimensional case, observability inequality is not equivalent to u.c.p. This gives different notions of controllability, namely exact and approximate. Indeed u.c.p is weaker than observability inequality.

**Theorem:** System (2.1) is exactly controllable in time T if and only if (2.3) is observable in time T.

**Proof (sketch):** Assume $(2.5)_b$. This implies the coercivity of $J$, i.e, $\lim_{|\phi_T|\to\infty} J(\phi_T) = \infty$. Continuity together with convexity of $J$, then implies the existence of a minimizer and hence Controllability. Conversely, if (2.1) is controllable and if $(2.5)_b$ is not true, then $\exists \phi_T^k \subseteq \mathbb{R}^n$, $k \geq 1$ such that $|\phi_T^k| = 1, \forall k$ and $\int_0^T |B^*\phi^k|^2 \to 0$ as $k \to \infty$ which implies that $\phi_T^k \to \phi_T$ (along a subsequence) and $|\phi_T| = 1$. Further, $\int_0^T |B^*\phi|^2 = 0$ where $\phi$ is the solution corresponding to $\phi_T$. From controllability, we have, $\exists u \in L^2(0,T)$ such that

$$\int_0^T \langle u, B^*\phi_k\rangle = -\langle x^0, \phi_k(0)\rangle, k \geq 1$$

Hence

$$\langle x^0, \phi(0)\rangle = 0 \Rightarrow \phi(0) = 0$$

As $x^0$ is arbitrary, we get $\phi_T = 0$ which is a contradiction because $|\phi_T| = 1$.

**Remark:** Thus the exact controllability problem reduces to

(i) an uncontrolled system (adjoint equation)

(ii) an observation and

(iii) an observability inequality.

**Kalman's Controllability Condition:** R.E.Kalman, in the 1960's gave an equivalent criteria for finite dimensional systems as: (2.1) is controllable if and only if Rank $[B, AB, ..., A^{n-1}B] = n$.

Of course, this is not generalizable to infinite dimensional systems, but we follow the path described earlier.

## 3. Interior Controllability of the Wave Equation

It is well known that wave equations models many physical phenomena such as small vibration of elastic bodies and propagation of sound. It is also important to note that it is a prototype for the class of hyperbolic equations possessing major properties of hyperbolic equations like *the lack of regularizing effects*, *finite speed of propagation* which have very important consequences in control theory.

We consider the problem (controlled system)

$$\left\{\begin{array}{c} y'' - \Delta y = u\chi_\omega \; in \; (0,T)\times\Omega \\ y = 0 \; on \; \sum = (0,T)\times\partial\Omega \\ y(0,\cdot) = y^0, y'(0,\cdot) = y^1 \end{array}\right. . \tag{3.1}$$

Here $y = y(x,t)$ is the state and the control $u = u(x,t)$ acts on a sub-region $\omega \subset \Omega, \Omega \subseteq \mathbb{R}^n$ is of class $C^2$, $T > 0$ and $\chi_\omega$ is the characteristic function of $\omega$.

**Existence and uniqueness:** For any $u \in L^2((0,T)\times\omega)$ and $(y^0, y^1) \in H_0^1(\Omega)\times L^2(\Omega)$, $\exists!$ weak solution $y$ such that $(y, y') \in C([0,T]; H_0^1(\Omega) \times L^2(\Omega))$ and is given by the variational formula

$$(y(t), y'(t)) = S(t)(y^0, y') + \int_0^t S(t-s)(0, u(s)\chi_\omega(s))ds \tag{3.2}$$

Here $S(t)$ is the group of isometries generated by the wave operator on $H_0^1 \times L^2$. Moreover, if $u \in W^{1,1}((0,T); L^2(\omega)), (y^0, y^1) \in (H^2 \cap H_0^1) \times H_0^1$, then $(y, y') \in C^1([0,T]; H_0^1 \times L^2) \cap C([0,T]; (H^2 \cap H_0^1) \times H_0^1)$

The wave equation is reversible in time. Hence, we can solve the equation backward in time with the initial condition $y(T, \cdot) = y_T^0$ and $y'(T, \cdot) = y_T^1$ for $0 \le t \le T$ (adjoint).

**Definition(Exact controllability):** We say (3.1) is exactly controllable in time T, if for every initial data $(y^0, y^1)$ and final data $(y_T^0, y_T^1) \in H_0^1 \times L^2, \exists$ a control $u \in L^2((0,T)\times\omega)$ such that the solution $y$ of (3.1) also satisfies $y(T, \cdot) = y_T^0, y'(T, \cdot) = y_T^1$.

**Definition(Null controllability):** The system (3.1) is said to be null controllable if $\exists u \in L^2((0,T)\times\omega)$ so that the solution $y$ of (3.1) satisfies $y(T) = 0 = y'(T)$.

**Remark**: Due to time reversibility, it is easy to see that the exact controllability and null controllability are equivalent (Exercise).

**Definition(Approximate controllability):** The system is approximately controllable if the reachable set $R(T)$ is dense in $H_0^1 \times L^2$, where
$R(T) = \{ (y(T), y'(T)) : y \text{ is a solution of (3.1)}, \ u \in L^2((0,T)\times\omega), \ (y^0, y^1) \in H_0^1 \times L^2 \}$.

**Remark:** By linearity the reachable set is convex. Since $\mathbb{R}^n$ is the only convex dense set in $\mathbb{R}^n$, approximate and exact controllability are the same in finite dimensional case.

**Variational approach and observability**

For $(\phi_T^0, \phi_T^1) \in L^2 \times H^{-1}(\Omega)$, consider the backward homogeneous equation

$$\begin{cases} \phi'' - \Delta\phi = 0 \ in \ (0,T)\times\Omega \\ \phi = 0 \ on \ \sum \\ \phi(T) = \phi_T^0, \phi'(T) = \phi_T^1 \end{cases} \tag{3.3}$$

We remark that, the data $(\phi_T^0, \phi_T^1)$ (given) here is much weaker (i.e, in a larger space) than the data in (3.1), one has to understand the solution using *transposition method* and

$\exists!$ solution $(\phi, \phi') \in C([0,T]; L^2(\Omega) \times H^{-1}(\Omega))$. Further it satisfies

$$\| \phi \|^2_{L^\infty(0,T;L^2)} + \| \phi' \|^2_{L^\infty(0,T;H^{-1})} \leq C \| (\phi^0_T, \phi^1_T) \|^2_{L^2 \times H^{-1}(\Omega)} .$$

One can follow a similar analysis as in the finite dimensional case. Multiply (3.1) by $\phi$, (3.3) by $y$, integrate by parts to get (Exercise):

**Proposition:** The control $u \in L^2((0,T) \times \omega)$ drives the initial data $(y^0, y^1) \in H^1_0 \times L^2$ to the final data $(0,0) = (y(T), y'(T))$ if and only if

$$\int_0^T \int_\omega \phi u = _{H^{-1}} \langle \phi'(0), y^0 \rangle_{H^1_0} - _{L^2} \langle \phi(0), y^1 \rangle_{L^2}$$

for all $(\phi^0_T, \phi^1_T) \in L^2 \times H^{-1}$ and $\phi$ is the solution of (3.3).

Note that R.H.S is the duality product between $L^2 \times H^{-1}$ and $H^1_0 \times L^2$ denoted by

$$\langle (\phi^0, \phi^1), (y^0, y^1) \rangle := _{H^{-1}} \langle \phi^1, y^0 \rangle_{H^1_0} - _{L^2} \langle \phi^0, y^1 \rangle_{L^2}$$

Due to the reversibility, it can also be stated that the system (3.1) is null controllable if $\exists u \in L^2\big((0,T) \times \omega\big)$ such that

$$\int_0^T \int_\omega \phi u = \langle (\phi^0, \phi^1), (y^0, y^1) \rangle \tag{3.4}$$

where $\phi$ is the solution of (3.3) with the initial condition $\phi(0) = \phi^0, \phi'(0) = \phi^1$ instead of the final condition.

The equation (3.4) is indeed the optimality condition for the minimization of the functional $J : L^2 \times H^{-1} \to \mathbb{R}$ defined by

$$J(\phi^0, \phi^1) = \frac{1}{2} \int_0^T \int_\omega |\phi|^2 + \langle (\phi^0, \phi^1), (y^0, y^1) \rangle$$

Suppose $(\hat{\phi}^0, \hat{\phi}^1)$ is a minimizer of $J$, then again from the first principle, it is easy to see that

$$\int_0^T \int_\omega \phi \hat{\phi} = \langle (\phi^0, \phi^1), (y^0, y^1) \rangle,$$

where $\hat{\phi}$ is the solution of (3.3) with the initial conditions $\hat{\phi}(0) = \hat{\phi}^0, \hat{\phi}'(0) = \hat{\phi}^1$. Comparing it with (3.4), we see that $u = \hat{\phi}|_\omega$ is a control which drives $(y^0, y^1)$ to $(0,0)$ in time $T$. Thus, we have a constructive procedure for getting controls.

Hence the problem of controllability reduces to the existence of a minimizer of $J$. Therefore the idea is to look for sufficient conditions for the existence of a minimizer. The following observability gives a sufficient condition for the existence of a minimizer.

**Observability Estimate (inequality):** Consider the equation

$$(3.5)\qquad \begin{cases} \phi'' - \Delta\phi = 0 \ in \ (0,T)\times\Omega \\ \phi = 0 \ on \ \sum \\ \phi(0) = \phi^0, \phi'(0) = \phi^1 \end{cases} .$$

**Definition:** The equation (3.5) is said to be *observable* in time T if $\exists$ a constant $C > 0$ such that

$$(3.6)\qquad \| (\phi^0, \phi^1) \|_{L^2\times H^{-1}} \leq C \int_0^T \int_\omega |\phi|^2 = C \| \phi \|^2_{L^2((0,T)\times\omega)}$$

for all $(\phi^0, \phi^1) \in L^2 \times H^{-1}$ and $\phi$ is the solution of (3.5). The inequality (3.6 ) indicates that by observing $\phi$ in a subset $\omega \subset \Omega$ for time upto $T$, one can completely (uniquely) determine the solution $\phi$ of (3.5).

**Remark:** The observability inequality is a sufficient condition for the controllability. For, if (3.6) is true, then the functional $J$ is coercive; i.e.

$$J(\phi^0, \phi^1) \to \infty \ as \ \| (\phi^0, \phi^1) \|_{L^2\times H^{-1}} \to \infty$$

Since $J$ is also convex and continuous (continuity follows from the estimate of $\phi$ in terms of the initial values), it follows that $J$ attains a minimum. This is from the standard calculus of variations.

**Theorem:** Let $K$ be a closed convex subset on a Hilbert space H and assume that $J : K \to \mathbb{R}$ is convex, lower semi-continuous and if $K$ is unbounded, assume further that $J$ is coercive $(that \ is \ J(x) \to \infty \ as \ \| x \| \to \infty)$. Then J attains a minimum in $K$.

**Remark:** The control given by this variational method has a minimal norm in $L^2((0,T)\times\omega)$ among all other controls if exist. To see this, let $\tilde{u}$ be any other control which drives the system to zero. If $u = \hat{\phi}|_\omega$ is the control given by the above variational method, then taking the test function $\phi = \hat{\phi}|_\omega$ and $u = \hat{\phi}|_\omega$ in the optimality condition, we get

$$\| \hat{\phi} \|^2_{L^2((0,T)\times\omega)} = \langle \hat{\phi}^1, y^0 \rangle_{1,-1} - \langle \hat{\phi}^0, y^1 \rangle_{2,2}$$

Again, for any other control $\tilde{u}$, we have

$$\int_0^T \int_\omega \hat{\phi}\tilde{u} = \langle \hat{\phi}^1, y^0 \rangle_{1,-1} - \langle \hat{\phi}^0, y^1 \rangle_{2,2}$$

Therefore

$$\| \hat{\phi} \|^2_{L^2((0,T)\times\omega)} = \int\int_\omega \hat{\phi}\tilde{u} \leq \| \hat{\phi} \| \| \tilde{u} \|$$

which implies

$$\| \hat{\phi} \|_{L^2((0,T)\times\omega)} \leq \| \tilde{u} \|_{L^2((0,T)\times\omega)}$$

**Remarks:** 1. We can also discuss the boundary controllability and derive analogous results. But due to time constraints, we discuss this problem while introducing *Hilbert Uniqueness Method (HUM).*

2. In the previous sections, we have reduced the problem of controllability to that of an observability inequality which requires for the proof of the existence of a minimizer. However, in general, the observability inequality need not hold for arbitrary $T$ or $\omega$. One requires that $T$ is sufficiently large (like the diameter of $\Omega$) and $\omega$ has to satisfy (or the part of the boundary where the controls are acting) certain geometric condition. C. Bardos, G. Lebean and J. Rauch has proved using micro local analysis that in the class of $C^\infty$ domains, the observability inequality holds if and only if $(\omega,\, T)$ satisfies certain geometric condition in $\Omega$: *Every ray of geometric optics that propagates in $\Omega$ and is reflected on its boundary $\Gamma$ enters $\omega$ in time less than $T$.* This makes, in general, $T$ should be greater than diameter $(\Omega)$. It is also shown later that geometric condition is sufficient even in the case of $C^3$ domains.

The other method uses multiplier techniques to prove the observability inequality which provides sufficient conditions. We mention more on these aspects during HUM discussion.

3. However, there is a nice way of proving observability inequality in 1D using Fourier expansion of solutions and Ingham's inequality.

4. We did not discuss approximate controllability and we are not planning to discuss in these lectures. We remark that the weak notion, namely the *unique continuation principle* can be used (enough) in this case.

5. Some remarks about the heat equation (parabolic case): The approximate controllability is a more suitable notion for the heat equation. This is a due to the smoothing effect of the heat equation. For $\Omega\backslash\omega \neq \phi, \omega \subset, \Omega$, we know that the solutions are $C^\infty(\Omega\backslash\omega)$. Hence if $y^1 \in R(T, y^0)$, the reachable set, then $y(T) = y^1|_{\Omega\backslash\omega}$ is $C^\infty$. So if we use the notion of exact controllability as $R(T, y^0) = L^2(\Omega)$, then the exact controllability will not hold for heat equation as the restrictions of $L^2(\Omega)$ functions to $\Omega\backslash\omega$ need not be smooth.

• But, then the other property, namely, *infinite speed propagation* helps to achieve the approximate controllability for any time $T > 0$.

• Even in the case of parabolic equation the variational approach can be employed to study the controllability problem (approximate, null) to that of an observability inequality for the adjoint equation.

However, due to the irreversibility of the heat equation, the observability inequality is much harder to prove. The multiplier technique do not apply. One of the important method in this direction is based on *Carleman inequalities*, which we will not discuss in these lectures.

But there are older methods based on wave or elliptic equations and its exact controllability. Again, there is a proof in 1D, which is based on moment problems.

## 4. 1D Wave equation and Ingham's inequality

**Theorem 4.1:** Let $(\lambda_n)_{n\in\mathbb{Z}}$ be a sequence of real numbers and $\gamma > 0$ such that

$$\lambda_{n+1} - \lambda_n \geq \gamma > 0,$$

for all $n$ and $T > \pi/\gamma$. Then $\exists C = C(T, \gamma) > 0$ such that for any finite sequence $(a_n)$, we have

$$\sum |a_n|^2 \leq C \int_{-T}^{T} \left| \sum a_n e^{i\lambda_n t} \right|^2 dt \tag{4.1}$$

The above inequality is basically, a generalization of classical Parseval's equality for orthogonal sequences.

**Remark:** Of course the reverse inequality is indeed true and in fact, it holds for any $T > 0$. But (4.1) is true if T is sufficiently large.

**Remark:** Note that $\gamma$ is the minimal gap between two consecutive elements. The following theorem asserts that we only requires the asymptotic distance $\gamma_\infty = \underline{\lim}_{n\to\infty} |\lambda_{n+1} - \lambda_n|$. This has an effect in the controllability result and provides an optimal T.

**Theorem 4.2:** Let $\lambda_n, \gamma, \gamma_\infty$ as above and $T > \pi/\gamma_\infty$ (Note that $\gamma_\infty \geq \gamma$ and hence $\pi/\gamma_\infty \leq \pi/\gamma$), then $\exists c_1, c_2 > 0$ such that for any finite sequence $(a_n)$, we have

$$c_1 \sum |a_n|^2 \leq \int_{-T}^{T} \left| \sum a_n e^{i\lambda_n t} \right|^2 \leq c_2 \sum |a_n|^2.$$

**Observability for ID wave equation:** Recall the problem

$$\begin{cases} y_{tt} - y_{xx} = u\chi_\omega \ in \ (t,x) \in \ I \times I \\ y(t,0) = y(t,1) = 0, \ t \in (0,T) \\ y(0) = y^0, y'(0) = y^1 \end{cases} . \tag{4.2}$$

Here $I = (0,1)$, $\omega = (a,b)$ an interval $\subset (0,1)$ where the controls are distributed. Observability inequality is

$$\| (\phi^0, \phi^1) \|^2_{L^2 \times H^{-1}} \leq C \int_0^T \int_a^b |\phi|^2 dx dt \tag{4.3}$$

where $\phi$ is the solution of

$$\begin{cases} \phi_{tt} - \phi_{xx} = 0 \ in \ I \times I \\ \phi(t,0) = \phi(t,1) = 0, t \in (0,T) \\ \phi(0) = \phi^0, \phi'(0) = \phi^1 \end{cases} . \tag{4.4}$$

**Spectral Decomposition:** The above equation can be written as $\Phi' + A\Phi = 0, \Phi(0) = \Phi^0$, where A is the unbounded operator on $H = L^2(0,1) \times H^{-1}(0,1)$ with $D(A) = H^1_0(0,1) \times L^2(0,1)$ defined by

$$A\Phi = \begin{pmatrix} -z \\ -\partial_x^2 \phi \end{pmatrix}, \ with \ \Phi = \begin{pmatrix} \phi \\ z \end{pmatrix}, z = \phi'$$

That is $A = \begin{pmatrix} 0 & -1 \\ -\partial_x^2 & 0 \end{pmatrix}$. The Laplace operator $-\partial_x^2$ is unbounded operator on $H^{-1}$ with domain $H^1_0$ defined as $\langle -\partial_x^2 \phi, \psi \rangle = \int \phi_x \psi_x, \phi, \psi \in H^1_0(0,1)$. In fact, A is an isomorphism from $H^1_0 \times L^2 \to (H^1_0 \times L^2)' = L^2 \times H^{-1}$.

• The eigenvalues and eigenfunctions of A are given by

$$\lambda_n = sgn(n)(n\pi i), n \in \mathbb{Z}^*$$

$$\Phi_n = \begin{pmatrix} 1/\lambda_n \\ -1 \end{pmatrix} \sin(n\pi x)$$

Further $\{\Phi_n\}$ is an orthonormal basis for $H^1_0 \times L^2$

• Since A is isomorphism, we also get $\{\lambda_n \Phi_n\} = \{A\Phi_n\}$ is an orthonormal basis for $L^2 \times H^{-1}$.

• $\Phi = \sum a_n \Phi_n \in H^1_0 \times L^2 \Leftrightarrow \sum |a_n|^2 < \infty$ and $\Phi = \sum a_n \Phi_n \in L^2 \times H^{-1} \Leftrightarrow \sum \frac{|a_n|^2}{|\lambda_n|^2} < \infty$.

Suppose that the initial data $(\phi^0, \phi^1)$ has the Fourier expansion $\begin{pmatrix} \phi^0 \\ \phi^1 \end{pmatrix} = \Phi^0 = \sum a_n \Phi_n \in L^2 \times H^{-1}$, then the solution $\Phi = \begin{pmatrix} \phi \\ \phi' \end{pmatrix}$ is given by $\Phi(t) = \sum a_n e^{\lambda_n t} \Phi_n$

Now let us see what to prove to get (4.3), namely the Observability inequality:

$$L.H.S = \| (\phi^0, \phi^1) \|_{L^2 \times H^{-1}} = \sum_{n \in Z^*} |a_n|^2 \frac{1}{n^2 \pi^2}$$

$$R.H.S = \int_0^T \int_a^b |\phi(t,x)|^2 = \int_a^b \int_0^T |\sum a_n e^{in\pi t} \frac{1}{n\pi} \sin n\pi x|^2$$

Apply Ingham's inequality for the sequence $\{n\pi\}$ and $\left\{\frac{a_n}{n\pi} \sin n\pi x\right\}$ to get (Note that $(n+1)\pi - n\pi = \pi := \gamma > 0$)

$$\begin{aligned} R.H.S &\geq \int_a^b \sum_{n\in\mathbb{Z}^*} \left| \frac{a_n}{n\pi} \sin n\pi x \right|^2 \; for \; T > \frac{2\pi}{\gamma} = 2 \\ &\geq \sum_{n\in\mathbb{Z}^*} \frac{|a_n|^2}{n^2\pi^2} \int_a^b \sin^2 n\pi x dx \end{aligned}$$

(Note,we have $T > 2\pi/\gamma$ since the interval is $(0,T)$ not $(-T,T)$). Now, if we show that $C = \inf_{n\in z^*} \underbrace{\int_a^b \sin^2 n\pi x}_{=C_n} > 0$, we get LHS(4.3)$\geq$ C RHS(4.3).

To see this,

$$\begin{aligned} C_n &= \int_a^b \sin^2 n\pi x = \int_a^b \frac{1-\cos 2n\pi x}{2} \\ &\geq \frac{b-a}{2} - \frac{1}{2|n|\pi} \end{aligned}$$

It follows that $\exists n_0$ such that $\inf_{n\geq n_0} C_n > 0$ since $b_n > 0, \forall n$, we get $\inf_n C_n > 0$. Thus we have the observability inequality for $T > 2$.

## 5. **Hilbert Uniqueness Method**

We take the case of boundary controllability. Of course, one can also work with interior controllability. Consider the problem

$$\text{(5.1)} \qquad \left\{ \begin{array}{l} y_{tt} - \Delta y = 0 \; in \; (0,T)\times\Omega = Q \\ y(0) = y^0, y'(0) = y^1 \; in \; \Omega \\ y = u \; on \; \Sigma = (0,T)\times\Gamma \end{array} \right. .$$

The control is acting through the boundary $\Gamma$ (or it can also be through a part of the boundary $\Gamma_0$) over the time 0 to T. We are looking for a control $u$ so that the solution $y$ satisfies $y(T) = y'(T) = 0$. That is, we are looking for null controllability.

**Motivation of the approach:** Let $U_{ad}$ be the set of all exact controls. That is $U_{ad} = \{u \in L^2(\Sigma) : y \; satisfies \; (5.1) \; and \; y(T) = y'(T) = 0\}$ The question is whether $U_{ad}$ non-empty or not? If $U_{ad}$ is non-empty, indeed, we would like to pick up the best control according to certain criteria. Let us begin by assuming that $U_{ad} \neq \phi$, that is there is a control and consider the problem of minimizing:

$$J(u) = \inf_{v \in U_{ad}} J(v), \; where \; J(v) = \frac{1}{2}\int_{\Sigma} v^2 \tag{5.2}$$

We look for the optimality system via the method of penalization (other method is duality). Let $\varepsilon > 0$ and consider the problem

$$\inf J_\varepsilon(v,z), where \; J_\varepsilon(v,z) = \frac{1}{2}\int_{\Sigma} v^2 + \frac{1}{2\varepsilon}\int_Q |z'' - \Delta z|^2 \tag{5.3}$$

where $z$ satisfies

$$\begin{cases} z'' - \Delta \; z \in L^2(Q) \\ z = v \; on \; \Sigma \\ z(0) = y^0, y'(0) = z^1, z(T) = z'(T) = 0 \end{cases} . \tag{5.4}$$

Note that, we are not demanding $z$ satisfies PDE and there will be many $z$ satisfying (5.4). Penalized problem will have a unique solution for each $\epsilon > 0$.

- Prove estimates on the solution independent of $\epsilon$
- Pass to the limit
- At the limit, we may have a solution to (5.2)

**Conclusion:** If $\exists$ one control, then $\exists$ a control with minimal $L^2$-norm. This allows us to define a map $(y^0, y^1) \to v = v(y^0, y^1)$ (control with minimal norm) which has stability properties and is continuous.

**Remark:** Of course this does not say, how to get the control. At this stage we write down the optimality system (observability) for (5.3) and then pass to the limit as $\epsilon \to 0$. The *Hilbert uniqueness method (HUM)* is based on these ideas.

**Optimality system for (5.3):** Assume $(u^\epsilon, y^\epsilon)$ be a solution of (5.3), i.e, $J_\epsilon(u^\epsilon, y^\epsilon) = \inf J_\epsilon(v,z)$. Then, one can see that

$$(5.5)\qquad \begin{cases} y^{\epsilon}_{tt} - \Delta y^{\epsilon} \in L^2(Q) \\ y^{\epsilon} = u^{\epsilon} \ on \ \Sigma \\ y^{\epsilon}(0) = y^0, y^{\epsilon'}(0) = y^1, y^{\epsilon}(T) = 0 = y^{\epsilon'}(T) \end{cases}.$$

and we have the optimality system for the co-state $p^{\epsilon}$ as

$$(5.6)\qquad \begin{cases} p_{\epsilon}'' - \Delta p_{\epsilon} = 0 \ in \ Q \\ p_{\epsilon} = 0 \ on \ \Sigma, \frac{\partial p_{\epsilon}}{\partial \nu} = u^{\epsilon} \ on \ \Sigma \end{cases}.$$

Further, one can check that $p_{\epsilon} = -\frac{1}{\epsilon}(y^{\epsilon''} - \Delta y^{\epsilon})$. Moreover, The limit equation is given by

$$\begin{cases} y'' - \Delta y = 0 \ in \ Q \\ y = u \ on \ \Sigma \\ y(T) = y'(T) = 0 \\ y(0) = y^0, y'(0) = y^1 \end{cases}.$$

and

$$\begin{cases} p'' - \Delta p = 0 \\ p = 0 \ on \ \Sigma \\ \frac{\partial p}{\partial \nu} = u \ on \ \Sigma \end{cases}.$$

This motivates to look for a control of the form $u = \frac{\partial p}{\partial \nu}$ and $p$ satisfies the equation $p'' - \Delta p = 0 \ in \ Q, p = 0 \ on \ \Sigma$. But what would be the initial condition?

**Hilbert Uniqueness method:** Thus, we start with arbitrary initial values $\{\phi^0, \phi^1\}$ and solve the problem

$$(5.7)\qquad \begin{cases} \phi'' - \Delta \phi = 0 \ in \ Q \\ \phi = 0 \ on \ \Sigma \\ \phi(0) = \phi^0, \ \phi'(0) = \phi^1 \end{cases}.$$

and then solve for $\psi$ as:

$$(5.7)_2\qquad \begin{cases} \psi'' - \Delta \psi = 0 \ in \ Q \\ \psi(T) = \psi'(T) = 0 \\ \psi = \frac{\partial \phi}{\partial \nu} \ on \ \Sigma \end{cases}.$$

Define a map $\wedge : (\phi^0, \phi^1) \mapsto (\psi(0), \psi'(0))$. We wish to find $(\phi^0, \phi^1)$ such that $\psi(0) = y^0, \psi'(0) = y^1$ so that the exact controllability is achieved and then the control is given by $\frac{\partial \phi}{\partial \nu}$ with the solution $y = \psi$. Enough to prove $\wedge$ is onto. We need appropriate spaces to define the solutions $\phi$ and $\psi$.

**Remarks:** Solution $\phi$ has finite energy, i.e.

$$E(t) = \frac{1}{2} \int_0^T \left( |\phi'|^2 + \frac{1}{2} |\nabla \phi(x,t)|^2 \right) dx < \infty$$

That is $\phi \in L^\infty(0,T; H_0^1(\Omega)),\ \phi' \in L^\infty(0,T; L^2(\Omega))$. Moreover, energy is conserved, i.e. for all $t$,

$$E(t) = E(0) = \frac{1}{2} \left( \int_0^T |\phi'|^2 + \int_0^T |\nabla \phi^0|^2 \right).$$

**Initial difficulties:** Let us mention two of the fundamental difficulties in the well-definedness of the above method before coming to the ontoness of $\wedge$.

i) For a.e. $t$, $\phi(t,\cdot) \in H^1(\Omega)$ and hence $\nabla \phi(t,\cdot) \in L^2(\Omega)$. Thus, in general $\frac{\partial \phi}{\partial \nu}\big|_\Sigma = \nabla \phi \cdot \nu \big|_\Sigma$ is not a well defined quantity as of now. In general, we may require $\phi(t,\cdot) \in H^2(\Omega)$ to define $\frac{\partial \phi}{\partial \nu}\big|_\Sigma$, which in general is not true. However, this difficulty is overcome by establishing, what is known as a *hidden regularity* for $\frac{\partial \phi}{\partial \nu}\big|_\Sigma$. In fact, $\frac{\partial \phi}{\partial \nu} \in L^2(\Sigma)$.

**Theorem:** For the finite energy solution $\phi$ of problem (5.7), the quantity $\partial_\nu \phi|_\Sigma = \frac{\partial \phi}{\partial \nu}\big|_\Sigma$ is in $L^2(\Sigma)$ and satisfies, for any $T > 0$:

$$\| \partial_\nu \phi \|^2_{L^2(\Sigma)} \leq C_T \left( \| \phi^0 \|^2_{H^1(\Omega)} + \| \phi^1 \|^2_{L^2(\Omega)} \right) \tag{5.8}$$

The proof is technical and long. It is based on the *multiplier method* with suitable multipliers. More precisely, the Rellich-Pohozev multipliers of the form $q_k(x) \frac{\partial \phi}{\partial x_k}$ are used, where $q = (q_1 \cdots q_n)$ is a smooth vector field and finally, choose q such that $q = \nu$ on $\Sigma$.

ii) The second problem is the interpretation of the solution $\psi$ with the weak Dirichlet data $\psi = \frac{\partial \phi}{\partial \nu}$ which is only in $L^2(\Sigma)$ by the previous theorem. The solution has to be interpreted with a weak $L^2$ boundary data. This is done using the *method of transposition* (duality, adjoint).

Given $f \in L^1(0,T; L^2(\Omega))$, $\theta^0 \in H_0^1(\Omega)$ and $\theta^1 \in L^2(\Omega)$, define the finite energy solution $\theta \in C\big([0,T]; H_0^1(\Omega)\big) \cap C^1\big([0,T]; L^2(\Omega)\big)$ of the system

$$(5.9) \qquad \begin{cases} \theta_{tt} - \Delta\theta = f \ in \ Q \\ \theta = 0 \ on \ \Sigma \\ \theta(0) = \theta^0, \ \theta'(0) = \theta^1 \end{cases} .$$

Multiplying $(5.7)_2$ by $\theta$ and (5.9) by $\psi$ (assuming there is a smooth solution $\psi$) and integrating by parts, we get

$$(5.10) \qquad \int_Q f\psi + \langle \theta^0, \psi'(0) \rangle + \langle \theta^1, \psi(0) \rangle = -\int \partial_\nu \theta \cdot \partial_\nu \phi$$

Indeed the last term is well defined due to hidden regularity.

**Definition(Transposition solution):** We say $\psi \in C([0,T]; L^2(\Omega)) \cap C^1\big([0,T]; H^{-1}(\Omega)\big)$ is a transposition solution of $(5.7)_2$ if (5.10 ) holds for all $f \in L^1(0,T; L^2(\Omega))$ and for all $(\theta^0, \theta^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

**Remark:** The unique existence can be proved using Riesz-representation theorem. Further $\psi$ satisfies the continuity estimate:

$$\| \psi \|_{L^\infty(0,T;L^2(\Omega))} + \| \psi' \|_{L^\infty(0,T;H^{-1}(\Omega))} \leq C\bigg( \| \phi^0 \|_{H_0^1} + \| \phi^1 \|_{L^2} \bigg)$$

Thus, we have $(\psi(0), \psi'(0)) \in L^2(\Omega) \times H^{-1}(\Omega)$. Now, define

$$\wedge : H_0^1(\Omega) \times L^2(\Omega) \to L^2(\Omega) \times H^{-1}(\Omega)$$

by $\wedge\big(\phi^0, \phi^1\big) = \big(\psi(0), -\psi'(0)\big)$. We easily get $\langle \wedge\big(\phi^0, \phi^1\big), (\phi^0, \phi^1) \rangle = \| \frac{\partial \phi}{\partial \nu} \|_{L^2(\Sigma)}^2$

We have the continuity of $\wedge$ by the estimate (5.8). To prove $\wedge$ is onto or an isomorphism, we need a *reverse inequality* of (5.8 ). In other words, we need to have

$$(5.11) \qquad \| (\phi^0, \phi^1) \|_{H_0^1 \times L^2}^2 \leq C \int_\Sigma |\partial_\nu \phi|^2$$

This is nothing but the observability inequality with the observation $\partial_\nu \phi$ at the boundary.

**Conclusion:** If (5.11 ) holds, then the controllability problem is solved. For, given $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, let $(\hat{\phi}^0, \hat{\phi}^1) \in H_0^1(\Omega) \times L^2(\Omega)$ solves $\wedge(\hat{\phi}^0, \hat{\phi}^1) = (y^0, -y^1)$

Now let $\hat{\phi}$ solves (5.7) with $\hat{\phi}(0) = \hat{\phi}^0, \hat{\phi}^1(0) = \hat{\phi}^1$ and solve $(5.7)_2$ for $\hat{\psi}$ with $\hat{\psi} = \frac{\partial \hat{\phi}}{\partial \nu}$ on $\Sigma$. Then by definition $\wedge(\phi^0, \phi^1) = (\hat{\psi}(0), -\hat{\psi}'(0))$ and thus $\hat{\psi}(0) = y^0, \hat{\psi}'(0) = y^1$. Hence the controllability problem (5.1) is solved with $y = \hat{\psi}$ with control $u = \frac{\partial \hat{\psi}}{\partial \nu}$.

**Remarks**

1. The method is constructive

2. If there is one control, then there will be many controls driving the system to rest at time $T$. But the control given by HUM is the best control in the sense that it is the minimal $L^2$ control.

Let us come back to the observability estimate. As remarked earlier, it will hold only if $T$ is sufficient large. This is due to the finite speed of propagation. The control acting on the boundary $\partial\Omega$ cannot transfer the information immediately to the interior of the domain. It takes time something in the order of the diameter of $\Omega$

Again the method of multipliers can be used to prove the following result.

**Observability Inequality:** Let $\Omega$ be of class $C^2$. Then $\exists T^0 > 0$ such that for $T > T_0$, the weak solution $\phi$ of (5.7) satisfies

$$(T - T_0) \parallel (\phi^0, \phi^1) \parallel^2_{H_0^1 \times L^2} \leq C \int\limits_{\Sigma} \left|\frac{\partial \phi}{\partial \nu}\right|^2$$

**Theorem:** $\exists T_0 > 0$ such that for $T > T_0$, the problem (5.1) is exactly controllable in time $T > T_0$.

**Remarks and comments:** One can get good estimates on the controllable time $T_0$. Also one need not have to apply control on the entire boundary $\Gamma$. But, at the same time, it is not possible to achieve controllability by taking arbitrary part of $\Gamma$ due to the geometric condition on $\Gamma$.

**A sufficient condition on the control part of $\Gamma$:** Let $x_0 \in \mathbb{R}^n$ be any fixed point and define $m(x) = x - x_0$. Define

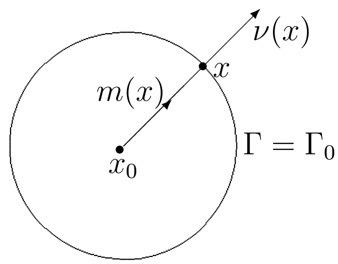
$$\begin{aligned}
\Gamma_0 : &= \{x \in \Gamma : m(x) \cdot \nu(x) \geq 0\}, \Gamma_1 = \Gamma \backslash \Gamma_0 \\
\Sigma_0 : &= (0, T) \times \Gamma_0, \Sigma_1 = (0, T) \times \Gamma_1 \\
R_0 : &= R(x_0) = \max_{x \in \bar{\Omega}} \{m(x)\}, T_0 = 2R(x_0)
\end{aligned}$$

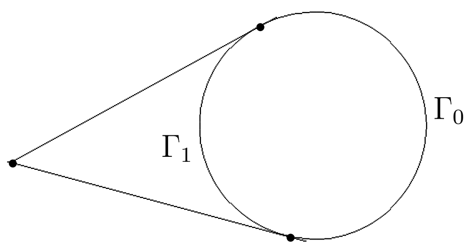
**Example:** If $\Omega$ is a ball of radius R, $x_0$ is the center, then $\Gamma_0 = \Gamma, R_0 = R, T_0 = 2R$. On the other hand, if $x_0$ is outside the ball $\Omega$, draw the tangents to the circle. Then $\Gamma_0$ is the arc that lies opposite to the point $x_0$.

One can prove the following observability estimate using the same multiplier technique.

**Theorem:** Let $T_0$ and $\Gamma_0$ be as above. Then, for $T > T_0$, the weak solution $\phi$ of (5.7) satisfies

$$(T - T_0) \parallel (\phi^0, \phi^1) \parallel^2_{H_0^1 \times L^2} \leq C \int_{\Sigma_0} \left|\frac{\partial \phi}{\partial \nu}\right|^2$$




**Remarks** 1. The above result gives us the option of acting control on certain parts of the boundary which clearly depends on the choice of $x_0$.

2. The minimum time required for controllability is greater than $T_0 = 2R_0 = 2R(x_0)$. Hence $T_0$ increases as $R_0$ increases. So our preference would be to choose the least $R_0$ which is the radius of the smallest circle containing $\Omega$ with center $x_0$. Thus the good choices of $x_0$ seems to be from $\Omega$. For example, if $\Omega$ is a ball, then the best choice of $x_0$ is the center and hence $T_0 = 2R$ =diameter($\Omega$).

There is a flip side to the story. Let us understand when $\Omega$ is a ball and $x_0$ is outside $\Omega$. Indeed $T > dia(\Omega)$ and hence we need a larger time. Then the advantage is that, we need not have to act on the entire boundary. If we think $x_0$ as an observer, then the control acts on that part of the boundary which the observer cannot see. As the point (observer) moves further away, one needs more and more time to achieve controllability but requires to apply the control on a shorter boundary, but always more than half of the boundary in the case of the circle (geometric condition).

**Generalization:** The HUM introduced is very general and can apply to many more systems. It can also apply to same system with different controllability spaces and different boundary condition. We sketch some of these aspects.

In the earlier situation, we had obtained the controllability in the space $L^2(\Omega) \times H^{-1}(\Omega)$ with controls in $L^2(\Sigma_0)$. In other words, the trajectories are moving in the space $L^2(\Omega) \times H^{-1}(\Omega)$.

Now consider, G as any Hilbert space of functions defined on $\Sigma_0$. Let $F_G$ be the completion of the space $D(\Omega) \times D(\Omega)$ with respect to the norm defined as $\| (\phi^0, \phi^1) \|_{F_G} := \| \frac{\partial \phi}{\partial \nu} \|_G$.

Recall that in the earlier situation, we have actually proved that $\| \frac{\partial \phi}{\partial \nu} \|_{L^2(\Sigma_0)}$ is an equivalent norm to $\| (\phi^0, \phi^1) \|_{H_0^1(\Omega) \times L^2(\Omega)}$. That is $G = L^2(\Sigma_0)$ and we established that $F_G = H_0^1(\Omega) \times L^2(\Omega)$ with the controllable space $L^2(\Omega) \times H^{-1}(\Omega) = F'_G$.

Introduce $\psi$ as the solution

$$\begin{cases} \psi'' - \Delta \psi = 0 \ in \ Q \\ \psi(T) = \psi'(T) = 0 \\ \psi = \begin{cases} I_G\left(\frac{\partial \phi}{\partial \nu}\right) \ on \ \Sigma_0 \\ 0 \ on \ \Sigma \backslash \Sigma_0, \end{cases} \end{cases} .$$

where $I_G : G \to G'$ is the canonical isomorphism. It is easy to see that

$$\langle \wedge(\phi^0, \phi^1), (\phi^0, \phi^1) \rangle = \| \frac{\partial \phi}{\partial \nu} \|_G^2$$

Consequently $\wedge : F_G \to F'_G$ is an onto isomorphism. Thus, for all $(y^0, -y^1) \in F'_G$, there exists a control $v \in G'$ such that $y = \psi$ satisfies $y(T) = y'(T) = 0$. Thus, we have the controllability in the space $F'_G$ with controls in $G'$.

Thus the crucial problem is the identification of $F_G$ and $F'_G$ which is not an easy task.

**Example:** Take $\| (\phi^0, \phi^1) \|_{F_1} := \left( \int_{\Sigma_0} \left| \frac{\partial \phi}{\partial \nu} \right|^p \right)^{1/p}, p > 1, p = 2$

In general, the characterization of $F_1$ is not known.

Now suppose $H$ is a linear operator defined on the function space of $\Sigma_0$. Further, suppose that the unique continuation principle holds: That is

$$H\left(\frac{\partial \phi}{\partial \nu}\right) = 0 \ on \ \Sigma_0 \Rightarrow \ \phi = 0 \ in \ Q.$$

In this case, $\| (\phi^0, \phi^1) \|_F = \| H(\frac{\partial \phi}{\partial \nu}) \|_G$ defines a norm on F and we have the controllability on the space $F'$ by HUM.

**Example:** Let $H = \frac{\partial}{\partial t}, G = L^2(\Sigma_0)$, $\| (\phi^0, \phi^1) \|_F := \| \frac{\partial \phi'}{\partial \nu} \|_{L^2(\Sigma_0)}$

In this case, we can identify $F, F'$ as $F = \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega)$ and $F' = H^{-1}(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega))'$ with the control space $(H^1(0,T; L^2(\Gamma_0))'$. Observe that the controllability is, indeed, achieved in a larger space than $L^2(\Omega) \times H^{-1}(\Omega)$, but the controls are in a weaker space $(H^1(0,T); L^2(\Gamma_0))'$ than $L^2(\Sigma_0)$. We skip the details.

**Remark:** We can also achieve controllability in a smaller space, namely $H_0^1(\Omega) \times L^2(\Omega)$ with better (smooth) controls, $v$ such that $v, \frac{\partial v}{\partial t} \in L^2(\Sigma_0)$. In fact, $v \in H_0^1(0,T; L^2(\Gamma_0))$.

This is quite easy, it is enough to work with the space $F = L^2(\Omega) \times H^{-1}(\Omega)$ for the initial values $(\phi^0, \phi^1)$ instead of $H_0^1(\Omega) \times L^2(\Omega)$ as done previously. Then, we get the controllability in the space $F' = H_0^1(\Omega) \times L^2(\Omega)$. Since $\phi^1 \in L^2(\Omega)$, let $\chi \in H_0^1(\Omega)$ be the solution of $\Delta\chi = \phi^1$ and define $\omega(t) = \int_0^t \phi(s)ds + \chi$ which satisfies

$$\begin{cases} w'' - \Delta w = 0 \ in \ Q \\ w(0) = \chi, w'(0) = \phi^0 \\ w = 0 \ on \ \Sigma \end{cases}.$$

By the hidden regularity, we see that $\frac{\partial w}{\partial \nu} \in L^2(\Sigma_0)$. Since $\phi = w'$, we get $\frac{\partial \phi}{\partial \nu} = \frac{\partial}{\partial t}\frac{\partial w}{\partial \nu} \in H^{-1}\big(0, T; L^2(\Gamma_0)\big)$

Thus the mapping $(\phi^0, \phi^1) \rightarrow \frac{\partial \phi}{\partial \nu}$ is linear continuous from $L^2(\Omega) \times H^{-1}(\Omega)$ to $H^{-1}(0, T; L^2(\Gamma_0))$. Hence by taking $G = H^{-1}(0, T; L^2(\Gamma_0))$, we get the control in the space $G' = H_0^1(0, T; L^2(\Sigma_0))$.

**Final Remark:** The HUM can be applied to many other situations; controllability with Neumann condition, more general elliptic operators, 4th order equations like Petrowski system etc.

## 6. Optimal Control

In the previous section, we have remarked that the control given by HUM minimizes the $L^2$ norm. We prove this fact in this section. Let

$$\mathcal{U}_{ad} := \{v \in L(\Sigma_0) : y(T, v) = y'(T, v) = 0, y \text{ is given by } (5.1)\} \tag{6.1}$$

That is $\mathcal{U}_{ad}$ is the set of all controls which steers the system to the origin. Note that $\mathcal{U}_{ad}$ is non-empty by the controllability results in the previous section. We are interested in minimizing the functional

$$J(v) := \frac{1}{2}\int_{\Sigma_0} |v|^2,$$

subject to $v \in \mathcal{U}_{ad}$. The main theorem is given below.

**Theorem:** Let $\{y^0, y^1\} \in Ł^2(\Omega) \times H^{-1}(\Omega)$ be the given initial values and let $u = \frac{\partial \phi}{\partial \nu}$ on $\Sigma_0$ is the control obtained by HUM. Then

$$J(u) = \min_{v \in \mathcal{U}_{ad}} J(v). \tag{6.2}$$

**PROOF:** We use the penalization method to prove the theorem (see [5]). Given $\varepsilon > 0$, introduce the functional

$$J_\varepsilon(v, z) = \frac{1}{2}\int_{\Sigma_0} |v|^2 + \frac{1}{2\varepsilon}\int_Q |z'' - \Delta z|^2, \tag{6.3}$$

where $v \in L^2(\Sigma_0)$ and $z$ is any solution of

$$
(6.4) \qquad \begin{cases} Lz \in L^2(Q), \\ z(0) = y^0, \;\; z'(0) = y^1, \;\; z(T) = z'(T) = 0 \;\text{ in }\; \Omega, \\ z = \begin{cases} v \;\text{ on }\; \Sigma_0, \\ 0 \;\text{ on }\; \Sigma_1. \end{cases} \end{cases}
$$

In particular, if we choose $v \in \mathcal{U}_{ad}$, then $z = y$ and $J_\varepsilon(v, z) = J(v)$. We now consider the optimal control problem

$$
(6.5) \qquad \inf\{J_\varepsilon(v, z) : \{v, z\} \;\text{ as in }\; (6.4)\}
$$

For fixed $\varepsilon > 0$, ( 6.5 ) is a standard optimization problem and it has a unique solution $\{u_\varepsilon, z_\varepsilon\}$ such that

$$
(6.6) \qquad J_\varepsilon(u_\varepsilon, z_\varepsilon) = \inf J_\varepsilon(v, z).
$$

**Claim:** The solution $\{u_\varepsilon, z_\varepsilon\}$ is bounded in $L^2(\Sigma_0) \times (L^\infty(0, T; L^2(\Omega)) \cap W^{1,\infty}(0, T; H^{-1}(\Omega)))$.

If $v \in \mathcal{U}_{ad}$ and $y = y(., v)$ is the corresponding solution of (5.1), then $\{v, y\}$ is a candidate for the minimization of ( 6.6 ) and hence

$$
J_\varepsilon(u_\varepsilon, z_\varepsilon) \leq J_\varepsilon(v, y(., v)) = J(v).
$$

Thus

$$
(6.7) \qquad J_\varepsilon(u_\varepsilon, z_\varepsilon) \leq \min_{v \in \mathcal{U}_{ad}} J(v).
$$

Hence $\{u_\varepsilon\}$ is bounded in $L^2(\Sigma_0)$. Moreover, if we put $f_\varepsilon = \frac{1}{\sqrt{\varepsilon}}(z''_\varepsilon - \Delta z_\varepsilon)$, then $\{f_\varepsilon\}$ is bounded in $L^2(Q)$ and hence

$$
(6.8) \qquad \|Lz_\varepsilon\| = \|z''_\varepsilon - \Delta z_\varepsilon\| \leq C\sqrt{\varepsilon}.
$$

From ( 6.4 ), ( 6.8 ) and from the estimates of the wave equation, it follows that $\{z_\varepsilon\}$ is bounded in $L^\infty(0, T; L^2(\Omega)) \cap W^{1,\infty}(0, T; H^{-1}(\Omega))$. Hence the claim.

From the claim, it follows that

$$
u_\varepsilon \to \hat{v} \;\text{ in }\; L^2(\Sigma_0) \;\; \textit{weak} \text{ and } z_\varepsilon \to \hat{y} \;\text{ in }\; L^2(Q) \;\; \textit{weak}.
$$

Further, $(\hat{v}, \hat{y})$ satisfies the system (5.1) by passing to the limit in ( 6.4 ), thanks to the estimate ( 6.8 ). Further, we have $\hat{y}(T) = \hat{y}'(T) = 0$ and therefore $\hat{v} \in \mathcal{U}_{ad}$. Now using the lower semi continuity and ( 6.7 ), we get

$$\min_{v\in\mathcal{U}_{ad}} J(v) \le J(\hat{v}) \le \underline{\lim} J(u_\varepsilon) \le \underline{\lim} J_\varepsilon(u_\varepsilon, z_\varepsilon) \le \min_{v\in\mathcal{U}_{ad}} J(v).$$

Thus

$$J(\hat{v}) = \lim\ J(u_\varepsilon) = \min_{v\in\mathcal{U}_{ad}} J(v), \tag{6.9}$$

which implies that $\hat{v}$ is an optimal control and

$$u_\varepsilon \to \hat{v} \ \text{ in } \ L^2(\Sigma_0) \ \ \textit{strong}. \tag{6.10}$$

**Claim:** $\hat{v} = \frac{\partial\phi}{\partial\nu}$.

Let $p_\varepsilon = \frac{1}{\varepsilon}(z''_\varepsilon - \delta z_\varepsilon) = \frac{1}{\sqrt{\varepsilon}} f_\varepsilon$, then we get the Euler equation of minimization of ( 6.6 ) as

$$\int_Q p_\varepsilon . L\zeta = \int_{\Sigma_0} u_\varepsilon v, \ \ \forall v \in L^2(\Sigma_0), \tag{6.11}$$

where, for given $v \in L^2(\Sigma_0)$, $\zeta$ is the solution of

$$\begin{cases} L\zeta \in L^2(Q), \\ \zeta(0) = \zeta'(0) = \zeta(T) = \zeta'(T) = 0 \ \text{ in } \ \Omega, \\ \zeta = \begin{cases} v \ \text{ on } \ \Sigma_0, \\ 0 \ \text{ on } \ \Sigma_1. \end{cases} \end{cases}$$

From ( 6.11 ), it follows that (integrating by parts)

$$\int_{\Sigma_0} u_\varepsilon v = \int_Q Lp_\varepsilon . \zeta - \int_\Sigma p_\varepsilon \frac{\partial\zeta}{\partial\nu} + \int_{\Sigma_0} \frac{\partial p_\varepsilon}{\partial\nu} v,$$

which is equivalent to

$$\begin{cases} Lp_\varepsilon = 0 \ \text{ in } \ Q, \\ p_\varepsilon = 0 \ \text{ in } \ \Sigma, \ \ \frac{\partial p_\varepsilon}{\partial\nu} = u_\varepsilon \ \text{ on } \ \Sigma_0. \end{cases}$$

Again, by the estimates of wave equation, we get

$$(T - T^0)(|\nabla p_\varepsilon(0)|^2 + |p'_\varepsilon(0)|^2) \le \frac{1}{2} T^0 \int_{\Sigma_0} |u_\varepsilon|^2 \le C, \ \ \forall \varepsilon > 0.$$

Thus the solution $p_\varepsilon$ is bounded in $L^\infty(0,T; H_0^1(\Omega)) \cap W^{1,\infty}(0,T; L^2(\Omega))$, so that

$$\begin{cases} p_\varepsilon \to p \ \text{ in } \ L^\infty(0,T; H_0^1(\Omega)) \ \ \textit{weak}\star, \\ p'_\varepsilon \to p' \ \text{ in } \ L^\infty(0,T; l2) \ \ \textit{weak}\star, \\ \{p_\varepsilon(0), p'_\varepsilon(0)\} \to \{p(0), p'(0)\} \ \text{ in } \ H_0^1(\Omega) \times L^2(\Omega) \ \ \textit{weak}. \end{cases}$$

Further $p$ satisfies

$$\begin{cases} Lp = 0 \text{ in } Q, \\ p = 0 \text{ in } \Sigma, \ \frac{\partial p}{\partial \nu} = \hat{v} \text{ on } \Sigma_0. \end{cases}$$

It follows from our observation that $(\hat{v}, \hat{y})$ satisfies (5.1) and from ( 6.12 ) that $\Lambda\{p(0), p'(0)\} = \{y^1, -y^0\}$. However, we already have, $\Lambda\{\phi^0, \phi^1\} = \{y^1, -y^0\}$. Since $\Lambda$ is an isomorphism, we must have $p(0) = \phi^0$ and $p'(0) = \phi^1$. Therefore by uniqueness of wave equation, $p = \phi$ and hence we get $\hat{v} = \frac{\partial p}{\partial \nu} = \frac{\partial \phi}{\partial \nu}$ on $\Sigma_0$. This completes the proof. □

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