Numerical Methods for PDE

In this chapter we discuss the finite difference methods for linear partial differential equations. We consider one example from each of Hyperbolic, Parabolic and Elliptic partial differential equations. The basic concept of consistency, stability and convergence of numerical schemes are discussed.

1 Hyperbolic equation

Simplest hyperbolic partial differential equation is one way wave equation,

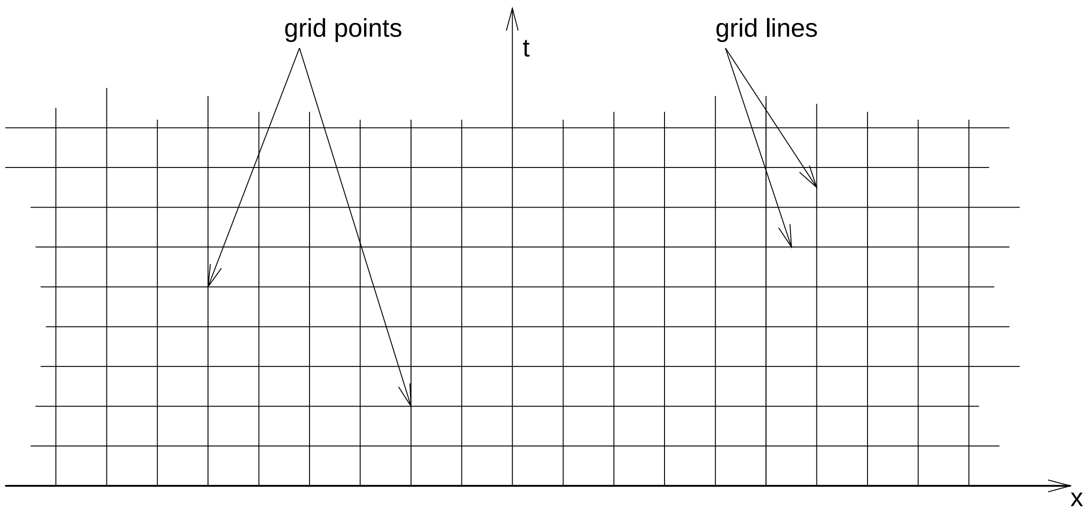
$$Pu = u_t + au_x = 0, \quad -\infty < x < \infty, t > 0, \quad u = u(x,t) \tag{1.1}$$

with initial condition

$$u(x,0) = u_0(x). \tag{1.2}$$

(1.1) - (1.2) is an initial value problem.

First define the grid of points in the (x,t) plane by drawing vertical and horizontal lines through the points (x_i, t_n)



where

$$\begin{aligned} t_n &= n\Delta t, \quad n = 0, 1, 2, \ldots \\ x_i &= ih, \quad i = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

The lines $x = x_i$ and $t = t_n$ are called grid lines and their intersections are called mesh points of the grid. We denote

$$u(x_i, t_n) = u(ih, n\Delta t) = u_i^n$$

The basic idea of **finite difference method is to replace derivatives by finite differences.** This can be done in many ways; as example we have

(i) Forward difference:

$$\frac{\partial u}{\partial t}(x_i, t_n) = \frac{u(x_i, t_{n+1}) - u(x_i, t_n)}{\Delta t} + O(\Delta t) = \frac{u_i^{n+1} - u_i^n}{\Delta t} + O(\Delta t)$$

(ii) Backward difference:

$$\frac{\partial u}{\partial t}(x_i, t_n) = \frac{u(x_i, t_n) - u(x_i, t_{n-1})}{\Delta t} + O(\Delta t) = \frac{u_i^n - u_i^{n-1}}{\Delta t} + O(\Delta t)$$

(iii) Central difference:

$$\frac{\partial u}{\partial t}(x_i, t_n) = \frac{u(x_i, t_{n+1}) - u(x_i, t_{n-1})}{2\Delta t} + O(\Delta t^2) = \frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} + O(\Delta t^2)$$

Similar formulas can be given for derivative with respect to x i.e.,

$$\begin{aligned} \frac{\partial u}{\partial x}(x_i, t_n) &= \frac{u_{i+1}^n - u_i^n}{h} + O(h) \ \text{(Forward difference)} \\ &= \frac{u_i^n - u_{i-1}^n}{h} + O(h) \ \text{(Backward difference)} \\ &= \frac{u_{i+1}^n - u_{i-1}^n}{2h} + O(h^2) \ \text{(Central difference)} \end{aligned}$$

By replacing the derivatives by finite differences and neglecting the error terms we have list of difference equations. For example

$$P_{\Delta t,h} v = \frac{v_i^{n+1} - v_i^n}{\Delta t} + a\,\frac{(v_{i+1}^n - v_i^n)}{h} = 0 \ \text{(forward time – forward space)} \tag{1.3}$$

$$P_{\Delta t,h} v = \frac{v_i^{n+1} - v_i^n}{\Delta t} + a\frac{(v_i^n - v_{i-1}^n)}{h} = 0 \ \text{(forward time – backward space)} \tag{1.4}$$

$$P_{\Delta t,h} v = \frac{v_i^{n+1} - v_i^n}{\Delta t} + a\frac{(v_{i+1}^n - v_{i-1}^n)}{2h} = 0 \ \text{(forward time – central space)} \tag{1.5}$$

$$P_{\Delta t,h} v = \frac{v_i^n - v_i^{n-1}}{\Delta t} + a\frac{(v_{i+1}^n - v_i^n)}{h} = 0 \ \text{(backward time – forward space)} \tag{1.6}$$

$$P_{\Delta t,h} v = \frac{v_i^n - v_i^{n-1}}{\Delta t} + a\frac{(v_i^n - v_{i-1}^n)}{h} = 0 \ \text{(backward time – backward space)} \tag{1.7}$$

Like this one can consider several schemes. Here in all we replace the derivatives by finite differences. Given a list of schemes one naturally ask the question which of the schemes are useful and which are not, as indeed some are not.

Example : Take $a = 1$ in (1.1)

$$u_t + u_x = 0$$

$$\text{with I.C}: u(x,0) = u_0(x) = \begin{cases} 1 & \text{if} \quad x \leq 0 \\ 2x^3 - 3x^2 + 1 & \text{if} \quad 0 \leq x \leq 1 \\ 0 & \quad x \geq 1 \end{cases}$$

Exact solution is given by

$$u(x,t) = u_0(x-t) = \begin{cases} 1 & x \leq t \\ 2(x-t)^3 - 3(x-t)^2 + 1 & 0 \leq x - t \leq 1 \\ 0 & x \geq t + 1 \end{cases}$$

Now consider the scheme (1.3) for this example, i.e.,

$$v_i^{n+1} = v_i^n - \frac{\Delta t}{h}(v_{i+1}^n - v_i^n) \text{ (forward time – forward space)}.$$

Let $\lambda = \frac{\Delta t}{h}$, then

$v_i^{n+1} = (1+\lambda)v_i^n - \lambda v_{i+1}^n$.

Let $x = 1$ and $x_{j_0} = 1$ for some $i = j_0$.

Then $v(1,0) = v(x_{j_0}, 0) = 0 = u(x_{j_0}, 0) = v_{j_0}^0$ and $v_j^0 = 0 \ \ j \geq j_0$,

$$\begin{aligned} v_{j_0}^1 &= (1+\lambda)v_{j_0}^0 - \lambda v_{j_0+1}^0 = 0 \\ &\vdots \\ v_{j_0}^n &= 0 \quad \forall\, n \text{ and for any choice of } \lambda \end{aligned}$$

$$\Rightarrow v_j^n = 0 \quad \forall j \geq j_0$$

$$\Rightarrow v(x,t) = 0 \quad \forall x \geq 1$$

Solution obtained from (1.3) does not converges to $u(x,t)$ as mesh size h and Δt goes to zero. Therefore Scheme (1.3) is not a correct scheme for $u_t + u_x = 0$. To study what are the schemes are useful (convergent) let us first introduce the concepts of consistency and stability.

Throughout the notes, norm $|| \cdot ||$ means either

$$||u|| = ||u||_2 = \left(h \sum_j |u(x_j)|^2\right)^{1/2} = \left(h \sum_j |u_j|^2\right)^{1/2}$$

or

$$||u|| = ||u||_\infty = \sup_j |u(x_j)| = \sup_j |u_j|.$$

Definition : Given a partial differential equation $Pu = 0$ and a finite difference scheme $P_{\Delta t,h}v = 0$ we say that the finite difference scheme is **consistent** with the partial differential equation in norm $||\cdot||$, if for the actual solution u of $Pu = 0$,

$$||P_{\Delta t,h}u|| \to 0 \quad \text{as} \quad \Delta t, h \to 0.$$

Definition : The finite difference method is accurate of order (p, q) in $||\cdot||$ if for the actual solution u of $Pu = 0$,

$$||P_{\Delta t,h}u|| = O(h^p) + O(\Delta t^q).$$

Example : Let u be solution of (1.1).Then

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} - \left(\frac{u_i^{n+1} - u_i^n}{\Delta t}\right) - a\left(\frac{u_{i+1}^n - u_i^n}{h}\right) + O(\Delta t) + O(h) = 0$$

i.e.,

$$||P_{\Delta t,h}u|| = O(\Delta t) + O(h).$$

Hence

$$P_{\Delta t,h}v = \frac{v_i^{n+1} - v_i^n}{\Delta t} + a\frac{(v_{i+1}^n - v_i^n)}{h} = 0 \text{ is consistent with } u_t + au_x = 0.$$

Similarly scheme (1.4), (1.5),(1.6) and (1.7) are consistent with $u_t + au_x = 0$.

In all these schemes one can write

$$v_i^{n+1} = \sum_{j=-k}^{k} \alpha_j v_{i+j}^n = \alpha_{-k}v_{i-k}^n + \dots \alpha_0 v_i^n + \dots + \alpha_k v_{i+k}^n .$$

By defining forward shift operator $S_+v_j = v_{j+1}$ and backward shift operator $S_-v_j = v_{j-1}$ we can write

$$v_i^{n+1} = \alpha_{-k}S_-^k v_i^n + \dots \alpha_{-1}S_-v_i^n + \alpha_0 v_i^n + \alpha_1 S_+ v_i^n + \dots + \alpha_k S_+^k v_i^n$$

where

$$S_\pm^k = S_\pm o S_\pm \dots o S_\pm (k \text{ times composition})$$

$$I = \text{ Identity operator } = S_-^0 = S_+^0.$$

Therefore

$$v_i^{n+1} = Q(S_+, S_-)v_i^n \qquad \forall\ i$$

In general one can write the one step scheme by

(1.7). $$v^{n+1} = Qv^n \text{ where } v^{n+1} = \left(\dots, v_{i-1}^{n+1}, v_i^{n+1}, v_{i+1}^{n+1}, \dots\right)^T .$$

and Q is a matrix.

Definition : The finite difference method (1.7) is called **stable** in $||\cdot||$, if there exist constants K and β such that

$$||v^n|| \leq Ke^{\beta t}||v^0||$$

where $t = n\Delta t, K$ and β are independent of h and Δt

Definition : A finite difference method is **unconditionally stable** if it is stable for any time step Δt and space step h.

Examples :

$$v_i^{n+1} = v_i^n - \frac{\Delta t}{h}(v_i^n - v_{i-1}^n)$$

a finite difference scheme for $u_t + u_x = 0$

$$v_i^{n+1} = v_i^n(1-\lambda) + \lambda v_{i-1}^n, \qquad \lambda = \frac{\Delta t}{h}$$

$$\begin{aligned} ||v^{n+1}||_\infty = \sup_i |v_i^{n+1}| &= \sup_i |v_i^n(1-\lambda) + \lambda v_{i-1}^n| \\ &\leq \sup_i \{|1-\lambda|\ |v_i^n| \ + \ \lambda\ |v_{i-1}^n|\} \ \leq |1-\lambda|\ ||v^n||_\infty + |\lambda|\ ||v^n||_\infty. \end{aligned}$$

If $0 < \lambda \leq 1$, then $||v^{n+1}||_\infty \leq ||v^n||_\infty \leq \ldots \leq ||v^0||_\infty$. Therefore this scheme is l_∞-stable if $\lambda \leq 1$ (conditionally stable).

Consider the following implicit scheme:
$v_i^n = v_i^{n-1} - \lambda(v_i^n - v_{i-1}^n)$ $n \geq 1$ (backward time - backward space)

i.e, $v_i^n(1+\lambda) - \lambda v_{i-1}^n = v_i^{n-1}$ This can be written in the matrix form, $Av^n = v^{n-1}$ where $A = (a_{ij})$ with $a_{ii} = (1+\lambda)$ and $a_{ii-1} = -\lambda$ and $a_{ij} = 0$ if $j \neq i, i-1$.

$A = (1+\lambda)[I + C]$ where $C = (C_{ij})$ with $C_{ii-1} = \frac{-\lambda}{1+\lambda}$ and $C_{ij} = 0$ if $j \neq i-1$. Hence $||C||_\infty \leq \frac{\lambda}{1+\lambda} < 1$. Hence A is invertible and

$$\begin{aligned} ||A^{-1}||_\infty &\leq \left(\frac{1}{1+\lambda}\right)\frac{1}{1 - ||C||_\infty} \\ &\leq \left(\frac{1}{\lambda+1}\right)\frac{1}{1 - \frac{\lambda}{1+\lambda}} \leq 1 \end{aligned}$$

Hence

$$v^n = A^{-1}v^{n-1} \quad , \quad \text{where } v^n = (\ldots, v_{i-1}^n, v_i^n, v_{i+1}^n \ldots)^T$$

$$||v^n||_\infty \leq ||A^{-1}||_\infty ||v^{n-1}||_\infty \leq ||v^{n-1}||_\infty \leq \ldots \leq ||v^0||_\infty$$

Hence this scheme is unconditionally stable.

Definition : A finite difference method is said to be **linear** if it is of the form

$$v_i^{n+1} = \sum_{j=-m_1}^{m_2} c_j v_{j+i}^n \text{ where } c_j\,'s \text{ are constants}$$

m_1, m_2 are non-negative integers.

Theorem (Lax): If a finite difference method is linear, stable and accurate of order (p,q) in $||\cdot||$, then it is convergent of order (p,q) in $||\cdot||$.

Proof :

$$\begin{aligned} v^n &= Qv^{n-1} \qquad\qquad Q = Q(S_+, S_-) \\ &= Q(Qv^{n-2}) \\ &\vdots \\ &= Q^n v^0, \qquad\qquad Q^j = Qo\ldots oQ(j \text{ times composition}) \end{aligned}$$

This implies

$$\begin{aligned}
||v^n|| &= ||Q^n v^0|| \leq K e^{\beta t} ||v^0|| \ \text{(Scheme is stable)} \\
\Rightarrow & \quad \frac{||Q^n v^0||}{||v^0||} \leq K e^{\beta t}
\end{aligned}$$

$$||Q^n|| = \sup_{||v^0|| \neq 0} \frac{||Q^n v^0||}{||v^0||} \leq K e^{\beta t}$$

Let $u(x,t)$ be the exact solution of the problem $Pu = 0$. Then

$$u^n = Q u^{n-1} + \Delta t (O(h^p) + O(\Delta t^q))$$

by the definition of accuracy of the scheme.

$$\begin{aligned}
w^n &= v^n - u^n, \qquad\qquad w^0 = v^0 - u^0 = 0 \\
w^n &= Q w^{n-1} + \Delta t (O(h^p) + O(\Delta t^q)) \\
&= Q^2 w^{n-2} + Q((O(h^p) + O(\Delta t^q))\Delta t) + \Delta t (O(h^p) + O(\Delta t^q)) \\
&\vdots \\
&= Q^n w^0 + \Delta t \sum_{j=0}^{n-1} Q^j (O(h^p) + O(\Delta t^q))
\end{aligned}$$

$$\begin{aligned}
||w^n|| &\leq \Delta t \sum_{j=0}^{n-1} ||Q^j|| (O(h^p) + O(\Delta t^q)) \\
&\leq (n+1) \Delta t K e^{\beta t} (O(h^p) + O(\Delta t^q)) \\
&\leq e^{t(\beta+1)} K (O(h^p) + O(\Delta t^q)) \\
&= O(h^p) + O(\Delta t^q).
\end{aligned}$$

This completes the proof.

Von Neumann Analysis

Let $v = (v_j)_{j=-\infty}^{\infty}$ be a sequence. Define the discrete Fourier transform of v by

$$\widehat{v}(\xi) = \sum_j v_j e^{ij\xi} \qquad i = \sqrt{-1}, \quad \xi \in [0, 2\pi).$$

Forward shift operator S_+ defined by

$$S_+ v = (S_+ v_j)_{j=-\infty}^{\infty}, \qquad S_+ v_j = v_{j+1}$$

Backward shift operator S_- defined by

$$S_-v = (S_-v_j)_{j=-\infty}^{\infty}, \qquad S_-v_j = v_{j-1}$$

$$\begin{aligned}
\widehat{S_+v} = \sum_j (S_+v_j)e^{ij\xi} &= \sum_j v_{j+1}e^{ij\xi} \\
&= \sum_j v_j e^{i(j-1)\xi} = e^{-i\xi}\sum_j v_j e^{ij\xi} \\
&\qquad = e^{-i\xi}\widehat{v}(\xi).
\end{aligned}$$

Similarly $\widehat{S_-v} = e^{i\xi}\widehat{v}(\xi)$.

Example :

$$u_t + u_x = 0$$

Consider the numerical scheme

$$\begin{aligned}
v_i^{n+1} &= v_i^n - \frac{\Delta t}{h}(v_i^n - v_{i-1}^n) \\
&= (1-\lambda)v_i^n + \lambda v_{i-1}^n, \qquad \lambda = \frac{\Delta t}{h} \\
&= (1-\lambda)v_i^n + \lambda S_- v_i^n \\
&= ((1-\lambda) + \lambda S_-)v_i^n \quad = Q(S_+, S_-)v_i^n
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \qquad v^{n+1} &= Q(S_+, S_-)v^n \\
\Rightarrow \qquad \widehat{v}^{n+1} &= \textstyle\sum_j v_j^{n+1}e^{ij\xi} \\
&= \textstyle\sum_j ((1-\lambda) + \lambda S_-)v_j^n e^{ij\xi} \\
&= \textstyle\sum_j (1-\lambda)e^{ij\xi}v_j^n + \sum_j \lambda e^{i\xi}v_j^n e^{ij\xi} \\
&= (1-\lambda)\widehat{v}^n + \lambda e^{i\xi}\widehat{v}^n = ((1-\lambda) + \lambda e^{i\xi})\widehat{v}^n \\
&= (1-\lambda+\lambda e^{i\xi})\widehat{v}^n
\end{aligned}$$

In general

$$\begin{aligned}
\widehat{v}^{n+1} &= Q(e^{-i\xi}, e^{i\xi})\widehat{v}^n \\
\rho(\xi) &= Q(e^{-i\xi}, e^{i\xi}) \text{ is called } \textbf{amplification factor}
\end{aligned}$$

Definition : A symbol $\rho(\xi)$ is said to satisfy the Von Neumann condition if there exists a constant $C > 0$ (independent of $\Delta t, h, n$ and ξ) such that

$$|\rho(\xi)| \le 1 + C\Delta t \qquad \text{for} \quad \xi \in [0, 2\pi)$$

Theorem : A finite difference method $v^{n+1} = Qv^n$ is stable in the l_2 norm iff the Von-Neumann condition is satisfied

Proof : Suppose the Von Neumann condition is satisfied. Let $v^{n+1} = Qv^n$. By Parsevel 's relation

$$\sum_j (v_j^{n+1})^2 = \frac{1}{2\pi}\int_0^{2\pi} |\widehat{v^{n+1}}(\xi)|^2 d\xi$$

upon multiplying by h

$$\begin{aligned}
||v^{n+1}||_2^2 &= \frac{h}{2\pi}\int_0^{2\pi} |\widehat{v}^{n+1}(\xi)|^2 d\xi \\
&= \frac{h}{2\pi}\int_0^{2\pi} |\rho(\xi)|^2 |\widehat{v}^{n}(\xi)|^2 \\
&\leq \frac{h}{2\pi}(1+C\Delta t)^2 \int_0^{2\pi} |\widehat{v}^{n}(\xi)|^2 = (1+C\Delta t)^2 ||v^n||_2^2
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
||v^{n+1}||_2 \leq (1+C\Delta t)||v^n||_2 & \\
\leq e^{C\Delta t}||v^n||_2 &\leq e^{C\Delta t}.e^{C\Delta t}||v^{n-1}||_2 \\
&\leq e^{2C\Delta t}||v^{n-1}||_2 \\
&\leq e^{(n+1)C\Delta t}||v^0||_2 \\
&\leq e^{2Ct}||v^0||_2
\end{aligned}$$

$\Rightarrow$ scheme is l_2 stable.

Conversely, suppose Von-Neumann condition is not satisfied. This implies for each $C > 0$ $\exists$ a number $\xi_C \in [0, 2\pi)$ such that $|\rho(\xi_C)| > (1 + C\Delta t)$. Since $\rho(\xi)$ is a continuous function of ξ there exists an interval $[\theta_1, \theta_2]$ in $[0, 2\pi)$ such that

$$|\rho(\xi)| > (1 + C\Delta t) \quad \forall \;\; \xi \in [\theta_1, \theta_2] = I_C$$

Consider the discrete initial data $v^0 = (v_j^0)$ such that

$$\widehat{v}(\xi) = \begin{cases} 0 & \text{if } \xi \notin I_C \\ \sqrt{\frac{2\pi}{h(\theta_2-\theta_1)}} & \text{if } \xi \in I_C \end{cases}$$

Then by Parsevel's relation

$$\begin{aligned}
||v^{n+1}||_2^2 = h\sum_j (v_j^{n+1})^2 &= \frac{h}{2\pi}\int_0^{2\pi} |\widehat{v}^{n+1}(\xi)|^2 d\xi \\
&= \frac{h}{2\pi}\int_0^{2\pi} |\rho(\xi)|^2 |\widehat{v}^{n}(\xi)|^2 d\xi \\
&= \frac{h}{2\pi}\int_0^{2\pi} |\rho(\xi)|^{2(n+1)} |\widehat{v}^{0}(\xi)|^2 d\xi \\
&= \frac{h}{2\pi}\int_{\theta_1}^{\theta_2} |\rho(\xi)|^{2(n+1)} |\widehat{v}^{0}(\xi)|^2 d\xi \\
&\geq (1+C\Delta t)^{2(n+1)}\frac{h}{2\pi}(\theta_2-\theta_1)\frac{2\pi}{h(\theta_2-\theta_1)} = (1+C\Delta t)^{2(n+1)} \\
&= (1+C\Delta t)^{2(n+1)}||v^0||_2^2
\end{aligned}$$

$\Rightarrow$

$$||v^{n+1}||_2 \geq (1 + C\Delta t)^{n+1} ||v^0||_2 \quad \text{for any } C > 0$$

$\Rightarrow$ scheme is not l_2 stable.

Examples : consider the equation

$$\begin{aligned} u_t + au_x &= 0 \\ u(x,0) &= u_0(x), \quad a \in I\!R^1 \end{aligned}$$

1. Godunov Scheme : It is given by

$$\begin{aligned} v_i^{n+1} &= v_i^n - \lambda \frac{(1 + sgn\ a)}{2} a(v_i^n - v_{i-1}^n) \\ &\quad - \lambda \frac{(1 - sgn\ a)}{2} a(v_{i+1}^n - v_i^n) \\ &= -a\lambda \frac{(1 - sgn\ a)}{2} v_{i+1}^n + (1 - \lambda a \frac{(1 + sgn\ a)}{2} + a\lambda \frac{(1 - sgn\ a)}{2}) v_i^n \\ &\quad + a\lambda \frac{(1 + sgn\ a)}{2} v_{i-1}^n \\ &= v_{i+1}^n \max(0, -a\lambda) + v_i^n (1 - |\lambda a|) + \max(0, a\lambda) v_{i-1}^n \ \text{ where } \lambda = \Delta t/h. \end{aligned}$$

l_∞ **stability:**

$$||v^{n+1}||_\infty = \sup_i |v_i^{n+1}| \leq \max(0, -a\lambda) \sup_i |v_{i+1}^n| + (|\ 1 - |\lambda a|\ |) \sup_i |v_i^n|$$

$$+ \max(0, a\lambda) \sup_i |v_{i-1}^n|$$

$$\begin{aligned} &\leq (\max(0, -a\lambda) + \max(0, a\lambda) + |\ 1 - |\lambda a|\ |) ||v^n||_\infty \\ &= (|a\lambda| + |\ 1 - |a\lambda|\ |) ||v^n||_\infty \end{aligned}$$

Godunov scheme is l_∞ stable if $|a\lambda| \leq 1$.

l_2 **stability:**

$$\rho(\xi) = e^{-i\xi} \max(0, -a\lambda) + (1 - |\lambda a|) + \max(0, a\lambda) e^{i\xi}$$

$$|\rho(\xi)| \leq |1 - |\lambda a|\ | + |a\lambda| \leq 1 \ \text{ if } \ |a\lambda| \leq 1$$

Von-Neumann condition is satisfied if $|a\lambda| \leq 1$

$\Rightarrow$ l_2-stable if $|\lambda a| \leq 1$

$\Rightarrow$ by Lax theorem scheme is convergent in l_2 norm and l_∞ norm if $|a\lambda| \leq 1$.

2. Lax-Friedrichs Scheme : It is defined by

$$v_i^{n+1} = \frac{v_{i+1}^n + v_{i-1}^n}{2} - \frac{\Delta t a}{2h} (v_{i+1}^n - v_{i-1}^n)\,.$$

Since

$$v_i^n = \frac{v_{i+1}^n + v_{i-1}^n}{2} + O(h^2),$$

$$\frac{v_{i+1}^n - v_{i-1}^n}{2h} = v_x + O(h^2),$$

we have

$$v_i^{n+1} = v_i^n - \frac{\Delta t a}{h}(v_{i+1}^n - v_{i-1}^n) + O(h^2)$$

$$\Rightarrow \frac{v_i^{n+1} - v_i^n}{\Delta t} - a\frac{(v_{i+1}^n - v_{i-1}^n)}{2h} + O\left(\frac{h^2}{\Delta t}\right) = 0$$

$$= v_t + O(\Delta t) + a v_x + O(h^2) + O\left(\frac{h^2}{\Delta t}\right)$$

$$= v_t + a v_x + O(\Delta t) + O(h) \qquad \text{if } \frac{h}{\Delta t} \text{ is bounded.}$$

The Lax-Friedrich's Scheme is first order accurate i.e. $p = 1, q = 1$

l_∞ **Stability:**

$$\begin{aligned} v_i^{n+1} &= \frac{(1-a\lambda)}{2}v_{i+1}^n + \frac{(1+a\lambda)}{2}v_{i-1}^n \\ ||v^{n+1}||_\infty = \sup_i |v_i^{n+1}| &\leq \frac{1}{2}|1-a\lambda| \sup_i |v_{i+1}^n| + \frac{1}{2}|1+a\lambda| \sup_i |v_{i-1}^n| \\ &\leq \frac{1}{2}(|1-a\lambda| + |1+a\lambda|)||v^n||_\infty \\ &\leq ||v^n||_\infty \quad \text{if} \quad |a\lambda| \leq 1 \end{aligned}$$

$\Rightarrow$ L. F. Scheme is l_∞ stable if $|a\lambda| \leq 1$

l_2 **Stability :**

$$\rho(\xi) = \frac{1}{2}(1-a\lambda)e^{-i\xi} + \frac{1}{2}(1+a\lambda)e^{i\xi}$$

$$|\rho(\xi)| \leq \frac{1}{2}|1-a\lambda| + \frac{1}{2}|1+a\lambda| \leq 1 \quad \text{if} \quad |a\lambda| \leq 1.$$

Hence L.F. Scheme is l_2 stable if $|a\lambda| \leq 1$.

L.F. Scheme is convergent in l_2 norm and l_∞ norm if $|a\lambda| \leq 1$.

3. Lax-Wendroff Scheme :

Let u be a solution of $u_t + a u_x = 0$

$$\begin{aligned} u(x, t+\Delta t) &= u(x,t) + \Delta t u_t(x,t) + \frac{(\Delta t)^2}{2}u_{tt}(x,t) + O(\Delta t^3) \\ &= u(x,t) - \Delta t a u_x + \frac{(\Delta t)^2}{2}a^2 u_{xx} + O(\Delta t)^3 \\ &= u(x,t) - \frac{\Delta t a}{2h}(u(x+h,t) - u(x-h,t)) + \frac{a^2(\Delta t)^2}{2h^2}(u(x+h,t) - 2u(x,t) \\ &\quad + u(x-h,t)) + \Delta t O(h^2) + (\Delta t)^2 O(h^2). \end{aligned}$$

Now Lax-Wendroff scheme is given by

$$v_i^{n+1} = v_i^n - \frac{\Delta t a}{2h}(v_{i+1}^n - v_{i-1}^n) + \frac{a^2(\Delta t)^2}{2h^2}(v_{i+1}^n - 2v_i^n + v_{i-1}^n)$$

This scheme is second order accurate i.e, $p = q = 2$

l_2 **stability :**

$$
\begin{aligned}
\rho(\xi) &= 1 - \frac{\lambda a}{2}(e^{-i\xi} - e^{i\xi}) + \frac{a^2\lambda^2}{2}(e^{-i\xi} - 2 + e^{i\xi}) \\
&= 1 + \frac{\lambda a}{2}(e^{i\xi} - e^{-i\xi}) + \frac{a^2\lambda^2}{2}(e^{i\xi} + e^{-i\xi} - 2) \\
&= 1 + \lambda a i \sin\xi + a^2\lambda^2(\cos\xi - 1) \\
&= 1 - a^2\lambda^2(1 - \cos\xi) + i\lambda a \sin\xi \\
&= 1 - 2a^2\lambda^2 \sin^2\frac{\xi}{2} + i\lambda a \sin\xi \ \text{ (because } 1 - \cos\xi = 2\sin^2\frac{\xi}{2})
\end{aligned}
$$

$$
|\rho(\xi)|^2 = 1 - 4a^2\lambda^2(1 - a^2\lambda^2)\sin^4\frac{\xi}{2}
$$

$$
|\rho(\xi)| \le 1 \quad \text{if} \quad |a\lambda| \le 1
$$

Scheme is l_2 stable. Hence converges in l_2 norm

Remark : Lax-Wendroff scheme is not l_∞ stable

4. Crank-Nicolson Scheme : Let u be the solution. Then

$$
\begin{aligned}
\frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} &= \frac{1}{\Delta t}\int_t^{t+\Delta t} u_t(x, \xi)d\xi \\
&= \frac{u_t(x, t + \Delta t) + u_t(x, t)}{2} + O(\Delta t^2) \\
&= -a\frac{u_x(x, t + \Delta t) + u_x(x, t)}{2} + O(\Delta t^2) \\
&= -\frac{a}{2}\frac{(u(x + h, t + \Delta t) - u(x - h, t + \Delta t))}{2h} \\
&\quad -\frac{a}{2}\frac{(u(x + h, t) - u(x - h, t))}{2h} + O(\Delta t^2) + O(h^2)
\end{aligned}
$$

Crank-Nicolson scheme is given by

$$
\frac{v_i^{n+1} - v_i^n}{\Delta t} = -\frac{a}{2}\frac{(v_{i+1}^{n+1} - v_{i-1}^{n+1})}{2h} - \frac{a}{2}\frac{(v_{i+1}^n - v_{i-1}^n)}{2h}
$$

This scheme is second order accurate i.e., $p = q = 2$

Let $\lambda = \frac{\Delta t}{h}$

$$
\begin{aligned}
-\frac{a\lambda}{4}v_{i-1}^{n+1} + v_i^{n+1} + \frac{a\lambda}{4}v_{i+1}^{n+1} &= v_i^n - \frac{a\lambda}{4}(v_{i+1}^n - v_{i-1}^n) \\
\left(\frac{-a\lambda}{4}S_- + I + \frac{a\lambda}{4}S_+\right)v_i^{n+1} &= \left(I - \frac{a\lambda}{4}(S_+ - S_-)\right)v_i^n \\
\left(-\frac{a\lambda}{4}e^{+i\xi} + 1 + \frac{a\lambda}{4}e^{-i\xi}\right)\widehat{v}^{n+1} &= \left(1 - \frac{a\lambda}{4}(e^{-i\xi} - e^{i\xi})\right)\widehat{v}^n \\
\widehat{v}^{n+1} &= \left(\frac{1 + \frac{a\lambda}{2}i\sin\xi}{1 - \frac{a\lambda}{2}i\sin\xi}\right)\widehat{v}^n \\
\rho(\xi) &= \frac{1 + \frac{a\lambda}{2}i\sin\xi}{1 - \frac{a\lambda}{2}i\sin\xi} = \frac{z}{\bar{z}} \\
|\rho(\xi)| &= 1
\end{aligned}
$$

Hence the Scheme is unconditionally stable.

$\Rightarrow$ Crank-Nicolson is convergent in l_2 norm

5. Unconditionally unstable scheme : Let a=1 in (1.1)

$$\begin{aligned} v_i^{n+1} &= v_i^n - \frac{\Delta t}{2h}(v_{i+1}^n - v_{i-1}^n) \\ \frac{v_i^{n+1} - v_i^n}{\Delta t} + \frac{v_{i+1}^n - v_{i-1}^n}{2h} &= v_t + v_x + O(\Delta t) + O(h^2), \end{aligned}$$

order of accuracy $= (p,q) = (2,1)$

$$\begin{aligned} \rho(\xi) &= 1 - \frac{\lambda}{2}(e^{-i\xi} - e^{i\xi}) \\ &= 1 + i\lambda \sin\xi \\ |\rho(\xi)|^2 &= 1 + \lambda^2 \sin^2\xi > 1 \text{ if } \xi \neq 0, \pi \end{aligned}$$

this scheme is not l_2 stable.

This scheme is not l_∞ stable

2 Parabolic equation

Consider the heat equation

$$u_t = bu_{xx} \qquad -\infty < x < \infty, \ t > 0, \ \ b > 0 \tag{2.1}$$

with initial condition

$$u(x,0) = u_0(x) \tag{2.2}$$

$$\begin{aligned} u_t &= \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t} + O(\Delta t) \\ u_{xx} &= \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} + O(h^2) \end{aligned}$$

The scheme for (2.1)

$$\frac{v_i^{n+1} - v_i^n}{\Delta t} = \frac{b}{h^2}(v_{i+1}^n - 2v_i^n + v_{i-1}^n) \text{ is of order } (p,q) = (2,1).$$

$$\begin{aligned} v_i^{n+1} &= v_i^n + \tfrac{\Delta t b}{h^2}(v_{i+1}^n - 2v_i^n + v_{i-1}^n) \\ &= v_i^n + \lambda b(v_{i+1}^n - 2v_i^n + v_{i-1}^n) \qquad \lambda = \frac{\Delta t}{h^2} \\ &= v_i^n(1 - 2\lambda b) + \lambda b v_{i+1}^n + \lambda b v_{i-1}^n \end{aligned} \tag{2.3}$$

l_∞ **Stability :**

$$\begin{aligned} ||v^{n+1}|| = \sup_i |v_i^{n+1}| &\leq (|1 - 2\lambda b| + \lambda b + \lambda b) \sup_i |v_i^n| \\ &\leq (|1 - 2\lambda b| + \lambda b + \lambda b)||v^n||_\infty \end{aligned}$$

if $\lambda b \leq 1/2$ then scheme is l_∞ stable

Define a scheme by

$$\begin{aligned}\frac{v_i^{n+1} - v_i^n}{\Delta t} &= \frac{b}{2}\frac{(v_{i+1}^{n+1} - 2v_i^{n+1} + v_{i-1}^{n+1})}{h^2} \\ &\quad + \frac{b}{2}\frac{(v_{i+1}^{n} - 2v_i^{n} + v_{i-1}^{n})}{h^2}\end{aligned}$$

This scheme is second order accurate i.e. $p = q = 2$.

$\Rightarrow$

$$v_i^{n+1} = v^n + \frac{b\Delta t}{2h^2}(v_{i+1}^{n+1} - 2v_i^{n+1} + v_{i-1}^{n+1}) + \frac{b\Delta t}{2h^2}(v_{i+1}^{n} - 2v_i^{n} + v_{i-1}^{n}) \tag{2.4}$$

The scheme (2.4) is called Crank-Nicolson Scheme.

<u>***θ*-Scheme :**</u> Let $0 \leq \theta \leq 1$, Define

$$v_i^{n+1} = v_i^n + \theta b\lambda(v_{i+1}^{n+1} - 2v_i^{n+1} + v_{i-1}^{n+1}) + (1-\theta)\lambda b(v_{i+1}^{n} - 2v_i^{n} + v_{i-1}^{n}) \tag{2.5}$$

If $\theta = 1/2$, θ - scheme is nothing but Crank-Nicolson Scheme.

Remark : The order of accuracy of the scheme (2.5)=$(p, q) = (2, 1)$ if $\theta \neq 1/2$. If $\theta = 1/2$ it is second order accurate i.e., p=q=2.

l_2 **stability of θ Scheme :** The scheme (2.5) can be written as

$$\begin{aligned}-\theta b\lambda v_{i+1}^{n+1} + (1+2\theta b\lambda)v_i^{n+1} - \theta b\lambda v_{i-1}^{n+1} &= (1-\theta)\lambda b v_{i-1}^n + (1 - 2(1-\theta)\lambda b)v_i^n \\ &\quad + (1-\theta)\lambda b v_{i+1}^n.\end{aligned} \tag{2.6}$$

Then

$$\begin{aligned}(-\theta b\lambda e^{-i\xi} + (1+2\theta b\lambda) - \theta b\lambda e^{i\xi})\widehat{v}^{n+1} &= \Big((1-\theta)\lambda b e^{i\xi} + (1 - 2(1-\theta)\lambda b) \\ &\quad + (1-\theta)\lambda b e^{-i\xi}\Big)\widehat{v}^n\end{aligned}$$

$$\widehat{v}^{n+1} = \frac{((1-\theta)\lambda b e^{i\xi} + (1 - 2(1-\theta)\lambda b) + (1-\theta)\lambda b e^{-i\xi})}{(-\theta b\lambda e^{i\xi} + (1+2\theta b\lambda) - \theta b\lambda e^{-i\xi})}\widehat{v}^n(\xi)$$

$$\rho(\xi) = \frac{(1-\theta)\lambda b(e^{i\xi} + e^{-i\xi}) + (1 - 2(1-\theta\lambda b)}{-\theta b\lambda(e^{i\xi} + e^{-i\xi}) + (1+2\theta b\lambda)}$$

$$\begin{aligned}&= \frac{2(1-\theta)\lambda b\cos\xi + (1 - 2(1-\theta)\lambda b)}{-2\theta b\lambda\cos\xi + (1+2\theta b\lambda)} \\ &= \frac{1 - 2(1-\theta)\lambda b(1-\cos\xi)}{1 + 2\theta b\lambda(1-\cos\xi)}\end{aligned}$$

Note that $1 - \cos\xi = 2\sin^2\frac{\xi}{2}$, hence

$$\rho(\xi) = \frac{1 - 4(1-\theta)\lambda b \sin^2 \frac{\xi}{2}}{1 + 4\theta b\lambda \sin^2 \frac{\xi}{2}}$$

Let $w = 4\lambda b \sin^2 \frac{\xi}{2}$ Now $-1 \leq \rho(\xi) \leq 1$

$$\Rightarrow \qquad -1 \leq \frac{1-(1-\theta)w}{1+\theta w} \leq 1$$

or $2 + 2\theta w \geq w \geq 0$,
$w \geq 0$ is obvious. Therefore

$$|\rho(\xi)| \leq 1 \quad \text{if} \quad (1-2\theta)w \leq 2 \tag{2.7}$$

If $\theta \geq 1/2$ (2.7) is obvious. For $0 \leq \theta < 1/2, (1-2\theta) > 0$. Hence $|\rho(\xi)| \leq 1$ if $(1-2\theta)4\lambda b \leq 2$. Therefore θ scheme is unconditionally stable if $\theta \geq 1/2$ and conditionally stable (i.e. $\lambda b \leq \frac{1}{2(1-2\theta)}$) for $0 \leq \theta < 1/2$.

l_∞ **Stability of θ-scheme :** Scheme (2.6) can be written as

$$A\, v^{n+1} = Bv^n$$

where $A = (a_{ij})$ is an infinite tridiagonal matrix with $a_{ii} = (1+2\theta\lambda b)$, $a_{ii-1} = a_{ii+1} = -\theta\lambda b$ and $a_{ij} = 0$ if $j \neq i, i-1, i+1$. $B = (b_{ij})$ is also an infinite matrix with $b_{ii} = (1-2(1-\theta)b\lambda)$, $b_{ii-1} = b_{ii+1} = (1-\theta)\lambda b$ and $b_{ij} = 0$ if $j \neq i, i-1, i+1$.

Now $A = (1+2\theta\lambda b)(I+C)$. Where $C = (C_{ij})$ be a matrix with $C_{ii+1} = C_{ii-1} = \frac{-\theta\lambda b}{1+2\theta\lambda b}$ and $C_{ij} = 0$ if $j \neq i-1, i+1$

$$||C||_\infty = \sup_i (\sum_j |C_{ij}|) = \frac{2\theta\lambda b}{1+2\theta\lambda b} < 1$$

Therefore $(I+C)$ is invertible and

$$||(I+C)^{-1}||_\infty \leq \frac{1}{1-||C||_\infty} = \frac{1}{1-\frac{2\theta\lambda b}{1+2\theta\lambda b}} = (1+2\theta\lambda b)$$

$$\Rightarrow \qquad ||A^{-1}||_\infty = \frac{1}{1+2\theta\lambda b} ||(I+C)^{-1}||_\infty \leq 1$$

Now $v^{n+1} = A^{-1}Bv^n$ and

$$||v^{n+1}||_\infty \leq ||A^{-1}||_\infty ||B||_\infty ||v^n|| \leq ||B||_\infty ||v^n||$$

Scheme is l_∞ stable if $||B||_\infty \leq 1$.

$$||B||_\infty = \sup_i (\sum_j |b_{ij}|) = 2(1-\theta)\lambda b + |(1-2(1-\theta)b\lambda|$$

If $1 - 2(1-\theta)b\lambda \geq 0$, then $||B||_\infty \leq 1$
θ-scheme is l_∞ stable if $b\lambda \leq \frac{1}{2(1-\theta)}$
θ scheme is convergent in l_∞ norm if $b\lambda \leq \frac{1}{2(1-\theta)}$

3 Elliptic equation

Consider the Dirichlet problem

$$P u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \text{ in } \Omega = (0,1) \times (0,1) \tag{3.1}$$

with boundary condition

$$u = g(x, y) \text{ on the boundary of } \Omega = \partial\Omega \tag{3.2}$$

Let us denote $u(x_i.y_j) = u_i^j, f(x_i, y_j) = f_i^j$and on boundary $u_i^j = v_i^j = g(x_i, y_j)$. Then the numerical scheme corresponding to (3.1) can be written as

$$P_{\Delta x, \Delta y} v = \frac{v_{i+1}^j - 2v_i^j + v_{i-1}^j}{(\Delta x)^2} + \frac{(v_i^{j+1} - 2v_i^j + v_i^{j-1})}{(\Delta y)^2} = f_i^j \tag{3.3}$$

which is second order accurate. Because if u is a solution of (3.1) then

$$P_{\Delta x, \Delta y} = O(\Delta x^2) + O(\Delta y^2)$$

Since the ratio of the mesh plays an insignificant role in the theory of elliptic problems. to study the above problem we take $\Delta x = \Delta y = h$ for simplicity. Then (3.3) becomes

$$v_{i+1}^j - 2v_i^j + v_{i-1}^j + (v_i^{j+1} - 2v_i^j + v_i^{j-1}) = h^2 f_i^j$$

i.e.,

$$4v_i^j = v_{i+1}^j + v_{i-1}^j + v_i^{j+1} + v_i^{j-1} - h^2 f_i^j, \quad 1 \le i, j \le M-1 \tag{3.4}$$

Let $x_0 = y_0 = 0$ $\quad x_i = i\Delta x = ih, i = 0, \ldots, M$
$x_M = y_M = 1$ $\quad y_i = i\Delta y = ih, i = 0, \ldots, M$
Let us consider the following simple cases to understand the scheme (3.4)

Let $M = 3$
Then by (3.4)

$$\begin{aligned}
4v_1^1 &- (v_2^1 + v_0^1 + v_1^2 + v_1^0) = -h^2 f_1^1 \\
4v_2^1 &- (v_3^1 + v_1^1 + v_2^2 + v_2^0) = -h^2 f_2^1 \\
4v_1^2 &- (v_2^2 + v_0^2 + v_1^3 + v_1^1) = -h^2 f_1^2 \\
4v_2^2 &- (v_3^2 + v_1^2 + v_2^3 + v_2^1) = -h^2 f_2^2
\end{aligned}$$

As on boundary $v_i^j = g_i^j$,this can be written as a linear system

$$Av = b_{g,f}$$

i.e.

$$Av = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} v_1^1 \\ v_2^1 \\ v_1^2 \\ v_2^2 \end{bmatrix} = \begin{bmatrix} g_0^1 & + & g_1^0 - h^2 f_1^1 \\ g_3^1 & + & g_2^0 - h^2 f_2^1 \\ g_0^2 & + & g_1^3 - h^2 f_1^2 \\ g_3^2 & + & g_2^3 - h^2 f_2^2 \end{bmatrix} = bg, f.$$

Now A can be written as

$$A = \begin{bmatrix} B & -I \\ -I & B \end{bmatrix}, \quad \text{where } B = \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A is a strictly diagonally dominant matrix. Hence A is invertible. Therefore above linear system can be solved uniquely.
Now let us consider the case $M = 4$

$$\begin{aligned}
4v_1^1 - v_2^1 - v_1^2 &= v_1^0 + v_0^1 - h^2 f_1^1 \\
4v_2^1 - v_3^1 - v_2^1 - v_1^1 &= v_2^0 - h^2 f_2^1 \\
4v_3^1 - v_3^2 - v_2^2 &= v_3^0 + v_4^1 - h^2 f_3^1 \\
4v_1^2 - v_2^2 - v_1^3 - v_1^1 &= v_0^2 - h^2 f_1^2 \\
4v_2^2 - v_3^2 - v_1^2 - v_2^3 - v_2^1 &= 0 - h^2 f_2^2 \\
4v_3^2 - v_2^2 - v_3^1 - v_3^3 &= v_4^2 - h^2 f_3^2 \\
4v_1^3 - v_2^3 - v_1^2 &= v_0^3 + v_1^4 - h^2 f_1^3 \\
4v_2^3 - v_3^3 - v_2^2 - v_1^3 &= v_2^4 - h^2 f_2^3 \\
4v_3^3 - v_2^3 - v_3^2 &= v_4^3 + v_3^4 - h^2 f_3^3
\end{aligned}$$

This can be written as a linear system

$$Av = b_{g,f}$$

i,e.,

$$Av = \begin{bmatrix}
4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4
\end{bmatrix}
\begin{bmatrix} v_1^1 \\ v_2^1 \\ v_3^1 \\ v_1^2 \\ v_2^2 \\ v_3^2 \\ v_1^3 \\ v_2^3 \\ v_3^3 \end{bmatrix} = b_{g,f}$$

Now A can be written as

$$A = \begin{bmatrix} B & -I & O \\ -I & B & -I \\ O & -I & B \end{bmatrix}, \quad \text{where } B = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$$

A is 9×9 matrix, B, O(zero matrix) and I are 3×3 matrices.
In general (3.4) can be written as

$$Av = b_{g,f}$$

where

$$A = \begin{bmatrix} B & -I & & & O \\ -I & B & -I & & \\ & -I & B & -I & \\ & & . & . & . \\ O & & & -I & B \end{bmatrix}$$

where

$$B = \begin{bmatrix} 4 & -1 & & O \\ -1 & 4 & -1 & \\ & . & . & . \\ O & & -1 & 4 \end{bmatrix}$$

with A is a matrix of order $(M-1)^2$. B , O and I are matrices of order $M-1$. v is a vector given by

$$v = (v_1^1, v_2^1, \ldots, v_{M-1}^1, v_1^2, v_2^2, \ldots, v_{M-1}^2, \ldots, v_1^{M-1}, \ldots, v_{M-1}^{M-1})^T$$

and b_g is a vector depends on the boundary values.

A is symmetric and positive definite. As A is tridioganal block matix there are several methods like direct methods or iterative methods to solve the above system.

Convergence : Let $u = u(x, y)$ be the actual solution of the problem and let $\epsilon_i^j = u_i^j - v_i^j$ where $u_i^j = u(x_i, y_j) = u(ih, jh)$ and v_i^j is obtained from (3.4). Since $\Delta u = f$ we have

$$4u_i^j - (u_{i+1}^j + u_{i-1}^j + u_i^{j+1} + u_i^{j-1} - h^2 f_i^j) + O(h^4) = 0$$

Therefore we have

$$L\epsilon = \epsilon_i^j - \frac{1}{4}(\epsilon_{i+1}^j + \epsilon_{i-1}^j + \epsilon_i^{j+1} + \epsilon_i^{j-1}) = O(h^4) \le Mh^4$$

for some $M > 0$. On the boundary $\epsilon_i^j = u_i^j - v_i^j \equiv 0$. Let $w(x, y) = x^2 + y^2$ and $w_i^j = w(ih, jh)$ Then

$$Lw = w_i^j - \frac{1}{4}(w_{i+1}^j + w_{i-1}^j + w_i^{j+1} + w_i^{j-1}) = -h^2$$

Define

$$\tilde{\epsilon}^j{}_i = \epsilon_i^j + Mh^2 w_i^j. \text{ Then}$$

$$L\tilde{\epsilon} = L\epsilon + Mh^2 Lw \le Mh^4 - Mh^4 = 0$$

$\Rightarrow$

$$\tilde{\epsilon}_i^j \le \frac{1}{4}(\tilde{\epsilon}_{i+1}^j + \tilde{\epsilon}_{i-1}^j + \tilde{\epsilon}_i^{j+1} + \epsilon_i^{j-1})$$

$\Rightarrow$ $\tilde{\epsilon}_i^j$ attains maxima on the boundary

Let r denotes the radius of a circle about the origin enclosing the region $\Omega = (0, 1) \times (0, 1)$. Then

$\tilde{\epsilon}_i^j \le$ maximum of ϵ_i^j on the boundary $+Mh^2r^2$

$= 0 + Mh^2 r = Mh^2 r^2$(because $\epsilon_i^j \equiv 0$ on boundary)

Now define $\underset{\sim}{\epsilon}_i^j = \epsilon_i^j - Mh^2 w_i^j$. By similar arguments one can show that

$$\underset{\sim}{\epsilon}_i^j \ge -Mr^2h^2$$

$$-Mr^2h^2 \le \underset{\sim}{\epsilon}_i^j \le \epsilon_i^j \le \tilde{\epsilon}_i^j \le Mh^2r^2$$

$$\Rightarrow |\epsilon_i^j| \le Mh^2r^2$$

Hence as mesh size goes to zero numerical solution goes to actual solution.

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APPENDIX:

1. Discrete Fourier Transform

Let $v = (v_j)_{j=-\infty}^{\infty}$ be a comlex sequence in l^2. Define the discrete Fourier transform of v by

$$\widehat{v}(\xi) = \sum_j v_j e^{ij\xi} \qquad i = \sqrt{-1}, \quad \xi \in [0, 2\pi)$$

. Then
(1)

$$v_j = \frac{1}{2\pi} \int_0^{2\pi} \widehat{v}(\xi)\, e^{-ij\xi} d\xi$$

(2) (Parsevel's relation)

$$\sum_j |v_j|^2 = \frac{1}{2\pi} \int_0^{2\pi} |\widehat{v}(\xi)|^2 d\xi$$

Proof: Let

$$L^2[0, 2\pi] = \{f : [0, 2\pi] \ \to C, \text{f is measureble and} \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 dx < \infty\}$$

Then $\{e_n = \frac{1}{\sqrt{2\pi}} e^{inx}\}_{n \in N}$ forms an orthonormal basis for $L^2[0, 2\pi]$. Therefore we have

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \widehat{v}(\xi) e^{-ij\xi} d\xi &= \frac{1}{2\pi} \int_0^{2\pi} (\sum_n v_n e^{in\xi}) e^{-ij\xi} \\ &= \frac{1}{2\pi} \sum_n v_n \int_0^{2\pi} e^{i(n-j)\xi}) d\,\xi \\ &= v_j. \end{aligned}$$

Also,

$$\frac{1}{2\pi} \int_0^{2\pi} |\widehat{v}(\xi)|^2 d\xi \ = \ \frac{1}{2\pi} \int_0^{2\pi} \widehat{v}(\xi)\, \overline{\widehat{v}}(\xi) d\xi$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_0^{2\pi} (\sum_n v_n e^{in\xi})\,(\overline{\sum_j v_j e^{ij\xi}})d\xi \\
&= \frac{1}{2\pi}\int_0^{2\pi} (\sum_n v_n e^{in\xi})\,(\sum_j \overline{v_j} e^{-ij\xi})d\xi \\
&= \sum_j |v_j|^2
\end{aligned}
$$