Hilbert functions and coefficients

Abstract

Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d and let J be an $\mathfrak{m}-$primary ideal. Let $Gr_J(A) = \bigoplus J^n/J^{n+1}$ be the associated graded ring of A with respect to the ideal J. The Hilbert-Samuel function of A with respect to J is $H_J(n) = \lambda(A/J^{n+1})$, (here $\lambda(-)$ denotes the length). It is well known that H_J is of polynomial type *i.e.* there exists $P_J(X) \in \mathbb{Q}[X]$ such that $H_J(n) = P_J(n)$ for all $n \gg 0$. We write

$$P_J(X) = e_0^J(A)\binom{x+d}{d} - e_1^J(A)\binom{x+d-1}{d-1} + \cdots + (-1)^d e_d^J(A).$$

Then the numbers $e_i^J(A)$ for $i = 0, 1, \cdots, d$ are the Hilbert coefficients of A with respect to J. The number $e_0^J(A)$ is called the multiplicity of A with respect to J.

Concerning first Hilbert coefficient it is a famous result of Northcott , which says that $e_1^J(A) \geq 0$. Precisely $e_1^J(A) \geq e_0^J(J) - \lambda(A/J)$.

It is clear that e_0 and e_1 are positive. As far as the higher Hilbert coefficients of J are concerned it is a famous result of Narita which says that $e_2^J(A) \geq 0$. In this case minmal value for $e_2^J(A)$ does not imply the Cohen-Macaulayness of $Gr_J(A)$. In the very same paper she also showed that if $\dim A = 2$, then $e_2^J(A) = 0$ if and only if J^n has reduction number one for some $n \gg 0$. In particular, $Gr_{J^n}(A)$ is Cohen-Macaulay.

Unfortunately, the well behaviour of the Hilbert coefficients stops with e_2. Narita showed that it is possible for e_3 to be negative. However, a remarkable result of Itoh says that if J is normal ideal then $e_3^J(A) \geq 0$.

In this seminar I would like to discuss our generalization of Northcott, Narita and Itoh results under a local homomorphism of Cohen-Macaulay local rings.