SOME PRELIMINARY NOTES ON HYPERBOLIC CONSERVATION LAWS

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1. Introduction

Many of the physical principles states that basic quantities like mass, energy,momentum,...are globally conserved. To formulate these laws in mathematical terms, we introduce some notations.

- Space variable : $x \in R^k$, Time variable : $t \in R$
- Densities of conserved quantities $u(x,t) \in \Omega$, which is an open connected subset of R^n .
- Flux : $f = (f_1, f_2, ...f_k)$ is the rate of flow across a surface per unit surface, each f_j is in general a function of x, t, u and its space derivatives and valued in R^n.
- Conservation laws says that : For any open bounded subset G of of space R^k with smooth boundary ∂G, rate of change of substance in G is equal to out flow of the substance from G.

$$\frac{d}{dt}\int_G u(x,t)dx = -\int_{\partial G} f.ndS$$

here $n : \partial G \to S^{k-1}$ is the outward unit vector and dS is the surface element on ∂G

Assuming u and f are smooth in their arguments we get

$$\frac{d}{dt}\int_G u(x,t)dx = \int_G \partial_t u(x,t)dx,$$

$$\int_{\partial G} f.ndS = \int_G \sum_{j=1}^{k} \partial_{x_j} f_j dx.$$

Here we used divergence theorem.

Then conservation law becomes, for all G open bounded subset of R^k

$$\int_G (\partial_t u + \sum_{j=1}^{k} \partial_{x_j} f_j)dx = 0$$

Now take $G = B_\epsilon(x) = \{y \in R^k : |x - y| < \epsilon\}$

$$\frac{1}{m_\epsilon(B(x)}\int_{B_\epsilon(x)} (\partial_t u + \sum_{j=1}^{k} \partial x_j f_j)dx = 0$$

Let ϵ go to 0, we get conservation law in PDE from :

$$\partial_t u + \sum_{j=1}^{k} \partial_{x_j} f_j = 0$$

- **Important remark :** To complete this theoretical formulation we need a law relating $(x, t, u, \partial_{x_j}, ...)$ to $f = (f_1, f_2, ...f_n)$.
- **Examples :**

1. Heat Conduction : Newton's law of cooling ; $f(u, \nabla u) = -h\nabla u$, h a positive constant, we get

$$u_t = h\triangle u$$

Next we give an example where the flux depends only on u
2.Traffic flow : $\rho(x, t)$ is density of car on a highway at a point x at time t.

Assume the speed of cars depends only on density $v = v(\rho)$,
flux f = number of cars crossing at the point x per unit time then

$$f = density \times velocity = \rho v(\rho).$$

Conservation law becomes $\frac{d}{dt}\int_a^b \rho(x, t)dx =$ [inflow at a -outflow at b] ie

$$\frac{d}{dt}\int_a^b \rho(x, t)dx = [\rho v(\rho)] - [\rho v(\rho)](b) = \int_a^b [\rho v(\rho)]dx$$

Conservation law becomes

$$\partial_t \rho + \partial_x(\rho v(\rho)$$

1.1. First order systems of conservation laws :

- In many physical phenomena in continuum mechanics, the flux depends only on u. In this case the conservation laws becomes first order systems :

$$\partial_t u + \sum_{j=1}^{k} \partial_{x_j} f_j(u) = 0$$

or more generally

$$\partial_t f_0(u) + \sum_{j=1}^{k} \partial_{x_j} f_j(u) = 0$$

where components of u are density, momentum, pressure, energy, etc.
- These systems appear as approximations of systems where effects of heat conduction, viscosity, capillarity,..etc are ignored. Mathematically this amounts to ignoring higher order derivative terms with small parameters which give smoothing effects. One of the most important such system is the following.

1. Compressible Euler equations in 3 -space dimension:

This is a 5×5 system describing conservation of mass momentum and energy:

$$\partial_t \rho + \sum_{j=1}^{3} \partial_{x_j}(\rho v_j) = 0$$

$$\partial_t(\rho v_i) + \sum_{j=1}^{3} \partial_{x_j}(\rho v_i v_j + \delta_{ij} p) = 0, i = 1, 2, 3$$

$$\partial_t(\rho E) + \sum_{j=}^{3} \partial_{x_j}(\rho v_j E + p v_j) = 0$$

Here ρ is the density, $v = (v_1, v_2, v_3)$ is velocity, p is pressure, $E = \frac{1}{2}|v|^2 + e$, $e = e(\rho, p)$ is given. δ_{ij} is Kroneckar delta.

This system has a long history which dates back to Euler [1775] contributions include Stokes, Riemann, Weyl, von Neumann.

Some simpler examples of conservation laws from continuum mechanics are given below.

2. Linearly degenerate system

$$u_t + Au_x = 0.$$

where A is a constant $n \times n$ strictly hyperbolic matrix. The simplest wave propagation problem and is linear.

3.Burgers equation

$$u_t + (u^2/2)_x = 0.$$

This is an example of genuinely nonlineay case.

4.Isentropic gas dynamics equation

$$\rho_t + m_x = 0, m_t + (m^2/\rho + p(\rho))_x = 0.$$

with , $m = u.\rho, p(\rho) = const.\rho^\gamma, \gamma > 1$, where ρ is density, u is velocity, $p(\rho)$ is pressure.

5.Equation of elasticity

$$u_t - \sigma(v)_x = 0, v_t - u_x = 0$$

u velocity, v deformation gradient, σ stress.

5. Tsunami waves : Under water disturbances such as volcanoes, earthquakes and landslides are the cause of tsunami waves some time reaching up to 40 meters. In the open ocean Tsunamis may be hard to spot. Long wave lengths can hide the size of the wave, Changes occur when the waves enter shallow water. The wave length shortens and height increase. Most of the time in deep ocean it moves as linear wave and at near the shore it is highly nonlinear. Modelling such waves is a very difficult problem.

1.2. **Initial value problem.** The Initial value problem that is to find u such that

$$\partial_t f_0(u) + \sum_{j=1}^{k} \partial_{x_j} f_j(u) = 0, x \in R^k, t > 0$$

with the initial conditions

$$u(x, 0) = u_o(x), x \in R^k$$

One of the basic question is global in time Well-posedness

- 1.Existence of solution in suitable function space

- 2.Uniqueness of solution
- 3.Continuous dependence of solution in suitable topology

Formation of singularities :

- Due to strong nonlinearity and absence of regularising effects, solutions which are initially smooth becomes discontinuous in finite time.
- No well developed theory for multidimensional systems of first order conservation laws.
- Solution space is not understood and interpretation of solution in some generalised sense is not clear. Even in the case weak formulation is available solutions are **NOT** unique .
- Selection principle for the physical solution is not well understood except for very special cases.
- Even for local wellposedness strong conditions on the system are required generally called hyperbolicity conditions.
- Right now we have a well developed theory for
  1. multidimensional scalar equations $(k \geq 1, n = 1)$
  2. systems of strictly hyperbolic conservation laws in one space variable $(k = 1, n \geq 1)$.

Summary of results for general symmetrizable systems

- Generally even local existence theory is not available for multidimensional systems.
- Friedrichs theory : Under reasonable conditions almost all first order systems of conservation laws of classical physics can be symmetryizable:

$$A_0 \partial_t u + \sum_{j=1}^{k} A_j(u) \partial_{x_j} u = 0$$

  Where $A_j, j = 0, 1, ...k$ are symmetric and A_0 is positive definite. This includes the compressible Euler system.
- Friedrichs : Linearized symmetric system is well posed in L^2 Sobolev spaces
- Lax : Nonlinear symmetric system is locally well posed in $H^s, s > \frac{n}{2} + 1$
- No global in time well posedness theory even for multidimensional symmetrizable systems .
- Structure of solutions are very complex. In addition to to shock waves, rarefaction waves and contact discontinuities, which are common in one space dimension, vorticity waves,focusing waves,concentration waves and there interactions makes makes the evolution complicated.

Main study of solutions of these systems are based on:

- computing by effective numerical methods, which itself is a challenging field.
- Asymptotic analysis/similarity reduction/special solutions
- Rigorous analysis of solutions and their qualitative properties is for two cases. 1. Multidimensional scalar equation- $n = 1, k \geq 1$. 2. Systems of hyperbolic conservation laws in one space variable-$k = 1, n \geq 1$

2. Systems of conservations in one space variable

First order systems of N conservation laws of the form

$$\partial_t u + \partial_x (f(u)) = 0, x \in R^1, t > 0 \tag{2.1}$$

for $u(x,t) \in \Omega$ appear in many physical applications. Here Ω is an open subset of R^N, and $f : \Omega \to R^N$ is a smooth function. The dependent variable u is called conservative variable and $f(u)$ is the flux associated with (2.1). The independent variables x and t are space and time respectively. In continuum physics (compressible fluid, nonlinear elastodynamics, phase transitions), equation of the form (2.1) represents fundamental principles of conservation of mass, momentum, energy, etc. when dissipative mechanisms are ignored. The system of equations (2.1) are approximations of physical models of the form

$$\partial_t u + \partial_x f(u) = (R(\epsilon u_x, \delta(\epsilon) u_{xx}, ...)_x \tag{2.2}$$

where ϵ is small and $R(\epsilon u_x, ...)$ contain higher order derivative terms of the unknown u. This equation carries small scale physical features (eg. viscosity and capillarity of the physical medium) and are neglected in the inviscid level in (2.1), by taking $\epsilon = 0$ and $R(0, 0, ...) = 0$ in (2.2).

2.1. Initial value problem for inviscid system.

Initial value problem is to find a function $u(x,t)$ of the system (2.1) with initial data

$$u(x, 0) = u_0(x) \tag{2.3}$$

in some suitable sense.

The questions that we are interested in are Well-posedness of the problem and qualitative properties of the solution operator $S_t u_0 = u(x,t)$.

- **Well-posedness in a suitable space :** Existence, Uniqueness, Continuous dependence on the data
- **Qualitaive properties of solution operator,** $S_t u_0(x) = u(x,t)$ **:** Smoothing properties, Compactness properties, Asymptotic large time behaviour, etc .
- **1. Existence :** When f **in nonlinear**, even for smooth initial data the solution (2.1) and (2.3) may develop discontinuities in finite time. To get global existence of solutions, one should work within a space of discontinuous functions and interpret solution of (2.1) in distributional sense:

$$\int_0^\infty \int_{-\infty}^\infty [u(x,t)\phi_t(x,t) + f(u(x,t)\phi_x(x,t)] dx dt + \int_{-\infty}^\infty u_0(x)\phi(x,0) dx = 0$$

for all $\mathbf{C}_0^\infty \left(R^1 \times [0,\infty)\right)$.

It is easy to see that if u is a C^1 solution then it is weak solution.

Weak formulation described above give very serious restrictions on the discontinuities on solutions :

Lemma : (Rankine-Hugoniot conditions) A piece wise smooth function u is a weak solution iff along any curve of discontinuity $x = \beta(t)$, the Rankine Hugoniot condition

$$-\frac{d\beta}{dt}[u(\beta(t)+, t) - u(\beta(t)-, t)] + [f(u(\beta(t)+, t)) - f(u(\beta(t)-, t)] = 0$$

is satisfied and in the region of smoothness the equation is satisfied in the classical sense.

Proof : Consider a curve of discontinuity and take point (x_0, t_0) on it. Take a nbd $B_\epsilon(x_0, t_0)$ of this point such that except along this curve the solution is smooth. Let D_- and D_+ are the points on this nbd which is left and right of this curve. By weak formulation, for $\phi \in C_c^\infty(B_\epsilon(x_0, t_0))$,

$$\int_D (u\phi_t + f(u)\phi_x)dxdt = \int_{D_+} (u\phi_t + f(u)\phi_x)dxdt + \int_{D_-} (u\phi_t + f(u)\phi_x)dxdt = 0.$$

By Green's theorem,

$$\begin{aligned}\int_{D_\pm} (u\phi_t + f(u)\phi_x)dxdt &= \int_{D_\pm} ((u\phi)_t + (f(u)\phi)_x)dxdt \\ &= \int_{\partial D_\pm} (-u\phi dx + f(u)\phi) \\ &= \int_{\partial D_\pm} (-udx + f(u)dt)\phi\end{aligned}$$

Using this and taking into account the orientation of integration, we get

$$\int \{-\frac{d\beta}{dt}[u(\beta(t)+, t) - u(\beta(t)-, t)] + [f(u(\beta(t)+, t)) - f(u(\beta(t)-, t)]\}\phi(\beta(t), t)dt = 0$$

for every $\phi \in C_c^\infty(B_\epsilon(x_0, t_0))$. This gives the R-H condition.

Remark : To verify u is a weak solution it is enough to check
(i). In the region of smoothness, $u_t + (f(u))_x = 0$ in the classical sense.
(ii). Along discontinuity curve, the R-H condition is satisfied.

- **2. Uniqueness:** Weak solutions are not unique, to choose the physical solution, we need to impose admissibility criteria on the solution. Inviscid system in itself is not complete. A well posed theory for the system cannot ignore the small scale features of a given physical problem.

- **3. Continuous dependence on data:** Need to identify the right function space and the right topology.
- **4. Qualitative properties :** Regularity, compactness and asymptotic behaviour, etc are related to irreversibility of the process
- In the case of scalar equations ie $n = 1$, there are several approaches. Calculus of variation and Hamilton Jacobi theory, nonlinear semi group theory, vanishing viscosity method, generalized characteristics, and various finite difference and finite volume approximations are some of them. We discuss some of them and theories of well posedness of the initial value problem in the space $BV \cap L^1$ with different types of admissibility criteria. We discuss some of them in detail.
- For $n = 1$ and $k = 1$ we have a complete theory starting with the work of Hopf [1950], and Lax[1957], which we will discuss in this course. This covers the case when the flux is convex. Krushkov [1970] developed a beautiful theory for scalar conservation laws for several space variables without any convexity conditions on the flux. We present that theory later in the course for one space dimensional case.

Now we discuss some generalities for systems and introduce some definition and notations.

2.2. **Strictly hyperbolic systems:** For system $n > 1$, when (2.1) is strictly hyperbolic, initial value problem is well studied starting with the work of Lax (1957) Glimm (1965), and Liu(1975).

- We say the system (2.1) is **strictly hyperbolic**, if the eigenvalues of the Jacobian matrix $Df(u)$ of the flux f are real and distinct eigenvalues :
$$\lambda_1(u) < \lambda_2(u) < ... < \lambda_N(u).$$
Then the corresponding left $\{l_j(u), j = 1, 2, ..N\}$ and right eigenvectors $\{r_j(u), j = 1, 2, ...N\}$ form a complete set for R^N for each $u \in \Omega$.
- We say the characteristic field λ_k is **genuinely nonlinear** in Ω if $\nabla\lambda_k(u).r_k(u) \neq 0$ for any $u \in \Omega$.
- We say λ_k is **linearly degenerate** in Ω if $\nabla\lambda_k(u).r_k(u) = 0$ for all $u \in \Omega$
- One way to construct solution to a general initial value problem (2.1) and (2.3) is using simpler problem called Riemann problem.

Definition: When initial data (2.3) takes a special form ie., consists of two constant states u_-, u_+ separated by a single discontinuity at $x = 0$:

$$u(x,0) = \begin{cases} u_-, & \text{if } x < 0, \\ u_+, & \text{if } x > 0 \end{cases} \tag{2.4}$$

the problem (2.1) and (2.4) is called the **Riemann problem.**

Remark : Plane waves play an important role in linear theory of hyperbolic equation because superposition of these special solutions gives solutions to general initial value problem. Even though in nonlinear case, linear superposition is not valid, special solutions given by the the Riemann problem is fundamental because they are the building block in the construction of solution of (2.1) with general initial conditions, for example using Glimm's scheme or front-tracking algorithm.

2.3. **Solution of Riemann problem.** Here we take the case when the system is strictly hyperbolic, and each characteristic field are either genuinely nonlinear or linearly degenerate.

Definition: A centered k - rarefaction wave is is a Lipschitz continuous solution of (2.1) of the form

$$u(x,t) = \begin{cases} u_-, & \text{if } x < \lambda_k(u_-t, \\ \Phi(x/t), & \text{if} \lambda_k(u_-)t < x < \lambda_k(u_+)t \\ u_+, & \text{if } x > \lambda_k(u_+t \end{cases}$$

connecting u_- to u_+ such that $\lambda_k(\Phi(\xi))$ increasing in ξ.

Definition : A centered k-shock is a solution of (2.1) with a discontinuity along a line $x = st$

$$u(x,t) = \begin{cases} u_-, & \text{if } x < st, \\ u_+, & \text{if } x > st \end{cases}$$

connecting u_- to u_+. We take the admissibility condition of Lax [1957]

$$\lambda_{k-1}(u_-) < s < \lambda_k(u_-), \lambda_k(u_+) < s < \lambda_{k+1}(u_+),$$

From the weak formulation, we have seen that the triple (s, u_-, u_+) must satisfy the Rankine-Hugoniot condition

$$f(u_+) - f(u_-) = s(u_+ - u_-).$$

Contact discontinuity : If λ_k is linearly degenerate a k-contact discontinuity is a solution of (2.1) with a discontinuity along a line $x = \lambda_k(u_-)t$:

$$u(x,t) = \begin{cases} u_-, & \text{if } x < \lambda_k(u_- t, \\ u_+, & \text{if } x > \lambda_k(u_- t. \end{cases}$$

Main question towards the construction of solution is the following. Fix u_-, what are the states u which can be connected to the right by a centered rarefaction wave, shock wave or contact discontinuity? The answer is given by the following **Theorem (Lax):** Suppose the k-th characteristic field is genuinely nonlinear in Ω and normalized so that $\nabla\lambda_k(u).r_k(u) = 1$. Let $u_- \in \Omega$. Then for $a > 0$ sufficiently small,

- there exists a one parameter family of states $u = u(\epsilon), 0 < \epsilon < a$, $u(0) = u_-$ (called **k- rarefaction wave curve**) which can connected to u_- from the right by a k- rarefaction wave. The parametrization can be chosen such that $u' = r_k, u'' = r'$, where $'$ denotes differentiation w.r.t ϵ
- There exists a one parameter family of states $u = u(\epsilon), -a < \epsilon < 0$, $u(0) = u_-$ (called **k -shock wave curve**)which can connected to u_- from the right by a k- shock wave with speed $s(\epsilon)$. The parametrization can be chosen such that $u'(0) = r_k(u_-, u''(0) = r'_k$, $s(0) = \lambda_k(u_-), s'(0) = 1/2$ where $'$ denotes differentiation w.r.t ϵ
- We denote $U_k(\epsilon)$ the combined shock - rarefaction wave curve passing through u_-, $\epsilon < 0$ corresponds to shock curve and $\epsilon > 0$ corresponds to rarefaction curve. The two curves have second order contact at $\epsilon = 0$

Suppose the k characteristic field is linearly degenerate.

- Then there exists a curve $u = U_k(\epsilon), |\epsilon| < a$ such that u_- can be connected to $U_k(\epsilon)$ by a contact discontinuity.

Theorem (Lax) : Given u_- in Ω, there exists a nbd $\Omega(u_-)$ such that for any u_+ in $\Omega(u_-)$ the Riemann problem (2.1) and (2.4) has a unique solution consisting of at most $(n+1)$ constant states separated by shocks, rarefaction or contact discontinuities.

Proof : For each $k, k = 1, 2, ...n$, there exists a one parameter family of transformation

$$T^k_{\epsilon_k} :\rightarrow R^n, |\epsilon_k| < a$$

which is a C^2 and any u in Ω can be connected to $T_{\epsilon_k} u$ on the right by a shock, rarefaction wave or a contact discontinuity.

$$U = \{(\epsilon_1, \epsilon_2, ...\epsilon_n) : |\epsilon_i| < a\}$$

Fix u_-. Consider the composite transformation $T : U \to \Omega_1(u_-$,$\Omega_1(u_-$ an open set subset of Ω containing u_-.

$$T(\epsilon) = T^n_{\epsilon_n} T^{(n-1)}_{\epsilon_{(n-1)}} ... T^1_{\epsilon_1}$$

Define $F(\epsilon) = T(\epsilon) - u_-$. We see that $F(0) = 0$ and $F(\epsilon) = \sum_{j=1}^n \epsilon_j r_j(u_-) + O(|\epsilon|^2)$. Thus Jacobian matrix of F at $\epsilon = 0$ is invertible. By inverse function theorem F is a diffeomorphism from a nbd of $0 \in U$ to a nbd of u_-. So for $|u_- - u_+|$ small there exists a unique $\epsilon = (\epsilon_1, ...\epsilon_n)$ such that $T(\epsilon) = u_+$.

Remarks on well-posed for strictly hyperbolic systems:

- Glimm (1965) proved a global existence theory using Glimm's scheme. Uniqueness and stability results were proved much later.
- By a series of papers Bressan, Liu and Yang (1999), Bressan, Crasta and Piccoli(2000) (see also some other references there) completed a wellposedness theory (existence, uniqueness and continuous dependence on initial data), for strictly hyperbolic systems with characteristic fields either genuinely nonlinear or linearly degenerate when the initial data are L^1 functions with small total variation.
- There are counter example which shows that for large data BV norm can blow up in finite time and hence a global existence theorem within the entire space of BV functions with bounded variation cannot hold in general.
- When systems are not strictly hyperbolic and/ or solutions with strong shock, the question of welpossedness is largely open.
- For general **strictly hyperbolic case** there are well posedness results if data is small in the space of BV functions based on a particular admissibility criteria. Although admissibility criteria are strongly interrelated but **they are not equivalent**, as we see later for the scalar case. There are may different selection principles and hence there are **many** theories of well-posedness and different theory may give different solution.

2.4. **Admissibility:** We have noted that from mathematical and physical reasons all types of discontinuous solutions are not admissible. The major question is to formulate this admissibility criteria and this is where **small scale features ignored while deriving the first order system (1) need to be taken into account in the formulation of well posedness theory**.

In many physical cases,(2.2) admits global in time smooth solution u^ϵ that are uniquely determined by their initial data. So a natural candidate for the inviscid system (2.1)is

$$u := \lim_{\epsilon \to 0} u^\epsilon$$

This selection principle is well known for scalar conservation laws and some special systems. For strictly hyperbolic case the analysis was carried out recently by Bianchini and Bressan (2005).

- In the construction of solution to Riemann problem, first construct k-wave curve, $k = 1, 2, ...n$ condition. The set of all states u which can be connected from a fixed state u_- by a k-rarefaction wave is a curve and those states which can be connected by a k-shock with admissibility is a also a curve and their union is called k-wave curve.

  If u_+ is in the k- wave curve through u_- for some k then we have a k-rarefaction or k-shock depending on the position of u_+ on the curve. Otherwise find $u_1, u_2, ...u_l$ on wave curves of different families so that one can connect u_- u_1, by a 1-wave u_1 to u_2 by a 2-wave $...u_l$to u_+ by an l-waves.
- **Remark .** Being constructive, this approach yields very detailed structure of solutions if successfully carried out. The difficulty is that it is hard to know the admissible shocks to enable this construction.

We list few of the commonly used criteria . Once we have this selection principle we can solve Riemann problem at least when the states u_- and u_+ are near by.

- **Viscosity admissibility criteria:** Viscosity admissibility criteria seeks to characterize admissible solutions of (1) as the $\epsilon \to 0$ limits of smooth solutions of parabolic systems

$$u_t + (f(u))_x = \epsilon u_{xx}$$

In the language of continuum physics, (works of Stokes(1848),Rankine(1870), Rayleigh(1911) in isothermal flows), admissibility is to be decided by visualizing the elastic medium as a limiting case in an appropriate class of media with internal dissipation.

This leads to traveling wave criteria for shock. A discontinuity of the form is admissible if the states $u_\pm$ can be connected by a traveling wave solution $u(x,t) = \Phi((x-st)/\epsilon)$ of the above parabolic system

$$-s\Phi' + f(\Phi)' = \Phi'', \Phi(\pm\infty) = u_\pm$$

- **Entropy admissibility condition:** This condition is due to Godunov(1961), Lax(1971) and seeks to get an admissibility directly through the system.

$$\eta(u)_t + (q(u))_x \leq 0$$

for pairs of functions η called entropy, q called entropy flux, satisfying $Dq(u) = D\eta(u).Df(u)$, η convex. The domain $(-\infty,\infty)\times[0,\infty)$ of any BV solution u of (2.1), may be decomposed into the union of 3 pairwise disjoint sets **C** the points of approximate continuity, **S** the points of approximate jump and a set **I** whose one-dimensional Hausdorff measure zero. With each point (x_0,t_0) on **S**, there is an associated s such that u attains distinct approximate limits u_- and u_+ on either side of the the line $x = x_0 + s(t-t_0)$ at (x_0,t_0) and the entropy condition reduces to

$$q(u_+) - q(u_-) - s(\eta(u_+) - \eta(u_-)) \leq 0$$

for points on the set **S**

- **Lax's Shock condition:**

An early example of a shock admissibility criterion in gas dynamics is that only compressive shocks are admissible. Riemann (1860) observed that this is equivalent to the requirement that shock be supersonic relative to the state at front and subsonic relative to the state on the back.

Lax(1957) fomulated this as a general shock condition,

$$\lambda_{k-1}(u_-) < s < \lambda_k(u_-), \lambda_k(u_+) < s < \lambda_{k+1}(u_+),$$

for genuinely nonlinear case. Liu (1975) extended this to a comprehensive shock admissibility criterion which work for more general characteristic fiels which is degenerate at isolated points.

All these are for waves of moderate strength in general.

- **Wave fan admissibility condition :**

This is due to Dafermos (1973) and is based on the observation that admissibility criteria should be compatible with translation and dilations of coordinates which leave the system invariant. So admissibility should be tested for Riemann problem. ie solutions of the form $u(x,t) = \Phi(x/t)$, which represents wave fans emanating from the origin at time $t = 0$. Wave fan admissibility criteria may be motivated by adding a variable viscosity to the inviscid system (1):

$$u_t + (f(u))_x = \epsilon t u_{xx}$$

which is invariant under dilation of co-ordinates. With Riemann type initial data (3), this reduces to

$$-y\Phi' + (f(\Phi))' = \epsilon\Phi''$$
$$\Phi(-\infty) = U_l, \Phi(\infty) = U_r$$

This is called **self similar viscosity approximation**.

- **Entropy rate admissibility criteria** says that a wave fan ie. a solution of the (1) and (3) of the form $u = \Phi(x/t)$ is admissible if $P_\Phi \leq P_{\Phi_1}$ for any other wave fan $u = \Phi_1(x/t)$ with the same end states as Φ. where

$$\begin{aligned} P_\Phi \quad &= \textstyle\sum_y q(\Phi(y+)) - q(\Phi(y-)) \\ &-y[\eta(\Phi(y+)) - \eta(\Phi(y-))] \end{aligned}$$

where the sum is over at most countable set of points y of jump discontinuity of Φ.

Remarks on wellposedness:

- We have listed few admissibility conditions among many. Viscous shock profile and the Lax/Liu- condition are sufficiently poweful to give uniqueness for strictly hyperbolic and shocks are of moderate strength.

3. Linearly degenerate case

First we consider linearly degenerate conservation laws. This is the case when $f(u) = Au$, A, a constant $n \times n$ matrix with of course strict hyperbolicity conditions and complete set of left and right eigenvectors as before but in the present case does not depend on the unknown $u(x,t)$. We consider initial value problem.

$$u_t + Au_x = 0$$

with initial condition

$$u(x,0) = u_0(x)$$

Any function $u(x,t)$ can be decomposed in the directions $r_j : u(x,t) = \sum_{j=1}^n c_j(x,t) r_j$ with $c_j(x,t) = l_j.u(x,t)$. It is easy to see that $u(x,t)$ is a solution of the system with initial conditions iff $c_j(x,t)$ solves

$$(c_j)_t + \lambda_j (c_j)_x = 0, \;\; c_j(x,0) = l_j.u_0(x).$$

By the method of characteristics we get its solution is

$$c_j(x,t) = l_j.u_0(x - \lambda_j t).$$

so that the solution of the system is given by

$$u(x,t) = \sum_{j=1}^{n} (l_j.u_0(x - \lambda_j t)) r_j.$$

Now if we look at the solution at (x,t), they are sum of jth components of signals at initial point $(y_j, 0)$ and propagating along the jth characteristics $x - \lambda_j t = y_j, j = 1, 2, ...n$.

Properties of solution operator $S_t u_0 = \sum_{j=1}^n (l_j.u_0(x - \lambda_j t)) r_j$

- It easy to see that if $u_0 \in C_B^k$ the space of k times continuously differential with bounded derivatives the solution also is in the same space, unique and depends continuously on the data.
- Solution is as smooth as initial data.
- For each $t > 0$, the shape of $l_j.u(.,t)$ is same as $l_j.u_0(x)$ except for translation of its position by $\lambda_j t$ and $\sup_{x\in R^1} |u(x,t)| = \sup_{x\in R^1} |u_0(x)|$ and so the solution operator $S_t u_0(x) = u(x,t)$ is invertible and in fact is unitary and $S_t : t \in R^1$ form a group
- Signals propagate along the characteristics $x = \lambda_j t + y$ and speed of propagation is finite and support spreads linearly.
- Singularities propagate along characteristics $x = \lambda_j t + y$

3.1. **Solution for the Riemann problem. Theorem :** The Solution of the Riemann problem for the linear system with left and right states, (u_-, u_+) is given by the following

$$u(x,t) = \begin{cases} u_- \ \text{ if } \ x < \lambda_1 t \\ \omega_i \text{if}, \ \ \lambda_i t < x < \lambda_{i+1} t, i = 1, 2, ...n-1, \\ u_+, \ \text{ if}, \ \ x > \lambda_n t \end{cases}$$

where $\omega_i = u_- + \sum_{j=i}^{i} l_j.(u_+ - u_-)$

Proof: Proof follows from the general solution obtained earlier, looking carefully the characteristic speeds.

Effects of viscosity.

$$u_t + Au_x = \frac{\epsilon}{2} u_{xx}$$

with initial condition

$$u(x,0) = u_0(x).$$

Here again we seek solution decomposed in the characteristic directions $u(x,t) = \sum_{j=1}^{n} c_j(x,t) r_j$ with $c_j(x,t) = l_j.u(x,t)$. It is easy to see that $u(x,t)$ is a solution of viscous system iff $c_j(x,t)$ solves

$$(c_j)_t + \lambda_j (c_j)_x = \frac{\epsilon}{2}(c_j)_{xx}, \ \ c_j(x,0) = l_j.u_0(x).$$

whose solution is

$$c_j(x,t) = \int_{R^1} l_j.u_0(y) d{\mu_{x,t}}^{\epsilon}(y).$$

where

$$d{\mu_{x,t}}^{\epsilon}(y) = \frac{1}{(2\pi t\epsilon)^{1/2}} e^{-\frac{(x-at-y)^2}{2t\epsilon}}$$

is the Gaussian measure on R^1. Note that the solution of (**??**) and (**??**) is given by

$$u^{\epsilon}(x,t) = \sum_{j=1}^{n} \int_{R^1} (l_j.u_0(y)) d{\mu_{x,t}}^{\epsilon}(y) r_j. \tag{3.1}$$

Even if $u_0(x)$ a bounded measurable function its solution given by (3.1) is infinitely differentiable solution.

Properties of solution operator $S_t^{\epsilon} u_0 = u^{\epsilon}(x,t)$

- The solution operator maps space of bounded measurable space to bounded C^∞ space.
- S_t^ϵ is compact and so not invertible. It is a smoothing operator.
- Speed of propagation is infinite.
- $\lim_{\epsilon\to 0} S_t^\epsilon u_0(x) = u(x,t)$ exits almost every where.
- $S_t^\epsilon \to \delta_{y=x-at}$ in the sense of measures as $\epsilon \to 0$. In particular if u_0 is bounded and continuously differentiable, $\lim_{\epsilon\to 0} S_t^\epsilon u_0(x) \to < \delta_{y=x-at}, u_0 > = u_0(x-at) = S_t u_0(x)$, which is the classical solution for the inviscid case.

This method of constructing solutions of first order equations from an approximation by diffusion term is generally called vanishing diffusion method. This method is particularly useful to treat non smooth solution of the first order equations.

Remarks : First we note that for each fixed $(x,t), t > 0$, the support of the measure $\mu^\epsilon_{(x,t)}(y)$ is R^1 where as the limiting measure is concentrated on the minimizer of

$$\min_{y\in R^1} \frac{(y-x+at)^2}{2t}.$$

We see a similar situation in nonlinear case (Burgers equation).

Weak solution : Assume u_0 is $BV(R^1)$ and $u_0^\epsilon = u_0 * e^\epsilon$, e^ϵ, being the standard Friedrichs mollifier. Let u^ϵ is the solution of the diffusion system with initial data u_0^ϵ. Then

$$\int_0^\infty \int_{-\infty}^\infty u^\epsilon \phi_t + A u^\epsilon \phi_x dxdt \quad + \int_{-\infty}^\infty u_0^\epsilon(x)\phi(x,0)dx = -\epsilon \int_0^\infty \int_{-\infty}^\infty u^\epsilon \phi_{xx} dxdt$$

for all $C_c(R^1 \times [0,\infty))$. Taking limits as $\epsilon \to 0$, we get

$$\int_0^\infty \int_{-\infty}^\infty u(x,t)\phi(x,t)_t + Au(x,t)\phi(x,t)_x dxdt + \int_{-\infty}^\infty u_0(x)\phi(x,0)dx = 0$$

for all $\phi \in C_c(R^1 \times [0,\infty))$.

Note that in the language of distribution theory,

$$u_t(\phi) = -\int\int u\phi_t dxdt, \;\; u_x(\phi) = -\int\int u\phi_x dxdt$$

for $\phi \in C_c(R^1 \times (0,\infty))$. So the above equation says

$$(u_t + Au_x)(\phi) = 0,$$

for all $\phi \in C_c(R^1 \times (0,\infty))$. In other words u is a distribution solution of $u_t + Au_x = 0$.

Remarks :

- In the passage to the limit as ϵ tend to 0, S_t^ϵ goes to S_t. S_t does not posses this compactness and the smoothing properties of the S_t^ϵ for $\epsilon > 0$.
- Indeed, S_t defined on the space of BV functions,

  $$S_t u_0 = \sum_{j=1}^n (l_j.u_0(x-\lambda_j t))r_j.$$

  defines a Unitary group and propagate signals and singularities along the characteristics. This is a linear phenomenon.
- However for nonlinear conservation laws and hyperbolic systems of equations nonlinearity preserves certain compactness and smoothing properties, in the passage to vanishing viscosity limit.

4. Burgers equation

We start with the inviscid Burgers equation which is a simple model for the nonlinear wave propagation phenomenon.

$$u_t + (\frac{u^2}{2})_x = 0 \tag{4.1}$$

with initial condition

$$u(x,0) = u_0(x). \tag{4.2}$$

The speed is $\lambda(u) = u$ and it depends on the unknown u and $\lambda'(u) = 1$ so the equation is genuinely nonlinear.

The equation says along the characteristic curve

$$\frac{dx}{dt} = u(x,t), x(0) = y, \quad \frac{du}{dt} = 0,$$

u is constant along the characteristics which is a straight line . Solving we get

$$x = u_0(y)t + y, u(x,t) = u_0(y)$$

From these relations we have $u(x,t) = u_0(x - u_0(y)t)$.

- This gives an implicit solution of the initial value problem
$$u(x,t) = u_0(x - u(x,t)t)$$
which can be solved at least for small time by the implicit function theorem.
- Another more useful way is to interpret the solution in the following way.
$$u(x,t) = \frac{x - y(x,t)}{t},$$
where $y = y(x,t)$ is a solution of the equation $x = u_0(y)t + y$, which exists for small $t > 0$.
- One important remark here is that $y(x,t)$ can be interpreted as the minimizer of
$$\min_{-\infty<y<\infty}[\int_0^y u_0(z) + \frac{(x-y)^2}{2t}]$$
because Lagrange equation is nothing but $x = u_0(y)t+y$. Minimizers $y(x,t)$ exists for each fixed (x,t) but may not be unique. We use this observation to construct global solution.

Non existence of smooth solutions : Note that unlike linear case here the characteristic speed depend on the unknown u.

- **Geometrical reason** is the characteristics may meet in finite time and then the solutions becomes multiple valued.
- **Analytical reason** is that while u remains bounded, its first order derivatives becomes infinity. Indeed
$$u_x = \frac{u_0'(y)}{1 + u_0'(y)t}$$
Now if $u_0'(x) < 0$ for some x, u_x blows up at $T = -\frac{1}{u_0'(y)}$, where y_0 is the point is where $u_0'(y)$ is minimum. At this point u_x becomes infinity. Thus u remains bounded but its first order derivatives becomes infinity in finite time.

For **Global solutions**, we use a formulation which does not require classical derivatives (distributions).

A locally integrable function u is said to be a weak solution of the (**??**) and (4.2) if

$$\int_0^\infty \int_{-\infty}^\infty u(x,t)\phi(x,t)_t + \frac{u^2(x,t)}{2}\phi(x,t)_x dxdt + \int_{-\infty}^\infty u_0(x)\phi(x,0)dx = 0$$

for all $\phi \in C_c(R^1 \times [0,\infty))$.

Entropy conditions : Solution in the weak solution is not unique. To find the unique physical solution we impose additional conditions. Here we discuss the following selection principles

- 1.Shock in equalities (stability reason)
- 2.Vanishing viscosity method (takes into account the small scale effects neglected in the inviscid level)
- 3.Mathematical entropy condition (additional conservation laws)

For scalar convex case all of them are equivalent for one space variable case.

4.1. **Hopf's work on Burgers equation :** Initial value problem is to find solution to (**??**) and (4.2), Hopf considered the equation with a viscous term

$$u_t + (\frac{u^2}{2})_x = \frac{\epsilon}{2}u_{xx}, \ \ u(x,0) = u_0(x) \tag{4.3}$$

and solved it explicitly.

Theorem : For fixed (x,t), $t > 0, x \in R^1$ and given bounded measurable function $u_0(x)$ define the measure $d\mu^\epsilon_{(x,t)}(y)$ defined by

$$d\mu^\epsilon_{(x,t)}(y) = \frac{e^{-\frac{1}{\epsilon}[\int_0^y u_0(z)dz + \frac{(x-y)^2}{2t}]}dy}{\int_{-\infty}^\infty e^{-\frac{1}{\epsilon}[\int_0^y u_0(z)dz + \frac{(x-y)^2}{2t}]}dy}$$

then an explicit solution of the initial value problem for Burgers equation with viscous term is given in the form

$$u^\epsilon(x,t) = \int_{R^n} \frac{(x-y)}{t} d\mu^\epsilon_{(x,t)}(y).$$

Proof : Hopf used the the Hopf-Cole transformation

$$u^\epsilon = -\epsilon \frac{v^\epsilon_x}{v^\epsilon}$$

and reduced the problem to the linear heat equation

$$v_t = \frac{\epsilon}{2}v_{xx}, v(x,0) = e^{-\frac{1}{\epsilon}\int_0^x u_0(z)dz}$$

Solving for v^ϵ and substituting in the Hopf-Cole transformation, we get the the explicit formula for u^ϵ as in the theorem.

The inviscid solution is constructed by passing to the limit as ϵ goes to zero.

Theorem: Assume u_0 is bounded measurable. For each fixed $t > 0$, except for a countable x, there exits a unique minimizer $y(x,t)$ for

$$\min_{-\infty<y<\infty}[\int_0^y u_0(z)dz + \frac{(x-y)^2}{2t}]$$

The measure associated with the viscous equation converges to a δ measure concentrated at this minimizer $y(x,t)$, namely for any continuous function $g(y)$ on R^1

$$\int g(y)d\mu^{\epsilon}_{(x,t)}(y) \to < \delta_{y(x,t)}, g(y)$$

The limit function

$$u(x,t) = < \delta_{y(x,t)}, \frac{(x-.)}{t} > = \frac{(x-y(x,t)}{t}$$

is well defined a.e. and BV function and solves the problem (**??**) and (4.2) in the weak sense. Further for each $t>0$, and $x \in R^1$, $u(x+,t)$ and $u(x-,t)$ exits and satisfies the entropy condition $u(x-,t) \geq u(x+,t)$.

Proof of the theorem: By an application of the theorem and Lemma in the appendix, we get

$$\lim_{\epsilon\to 0} u^{\epsilon}(x,t) = \frac{(x-y(x,t)}{t}, \quad a.e(x,t)$$

Now to show that the solution satisfies the equation in weak sense,

$$\int_0^{\infty}\int_{-\infty}^{\infty}(u^{\epsilon}\phi_t + \frac{(u^{\epsilon})^2}{2}\phi_x)dxdt + \int_{-\infty}^{\infty} u_0(x)\phi(x,0)dx = -\frac{\epsilon}{2}\int_0^{\infty}\int_{-\infty}^{\infty} u^{\epsilon}(x,t)dxdt$$

Using the fact that u^{ϵ} is bounded, an application of dominated convergence theorem gives the result.

To show u satisfies the entropy in equality, we just observe that $y_-(x,t) \leq y_+(x,t)$, so that

$$u(x-,t) = \frac{x-y_-(x,t)}{t} \leq \frac{x-y_+(x+,t}{t}$$

Remarks

- Here we remark that a little computation shows that viscous solution can be written in the form
$$u^{\epsilon}(x,t) = \int_{R^n} u_0(y)d\mu^{\epsilon}_{(x,t)}(y).$$
and if u_0 is Lipschitz continuous, then the entropy solution can also be written as
$$u(x,t) = < \delta_{y(x,t)}, u_0(y) >$$
- With this it easily follows that Hopf's formula gives an extension of the solution obtained by the method of characteristic for smooth region to non-smooth region as the derivative condition for the minimizer is the same as the equation of the characteristic $x = u_0(y)t + y$.
- $$S^{\epsilon}_t u_0 = < \mu^{\epsilon}_{(x,t)}(y), u_0 >$$
defines a semigroup, compact and hence non-invertible. It easily follows that
$$S_t u_0 = < \delta_{y(x,t)}, u_0 >$$
defines a semigroup, compact and hence non-invertible.
- Compactness is a property usually associated with parabolic equations. Unlike in the linear case, this property is preserved as $\epsilon \to 0$. The reason for this is the rate of energy dissipation does not tend to zero as $\epsilon \to 0$, in the nonlinear case where as for linear case it goes to zero.(This can be easily seen by following intuitive argument $\frac{d}{dt}\int (u^{\epsilon})^2 dx = -\epsilon \int |u^{\epsilon}_x|^2 dx$. Shock layer

has thickness ϵ and in the layer shape is traveling wave $w^\epsilon(x,t) = w(\frac{x-st}{\epsilon})$ where as for the linear case it is transition layer $w^\epsilon(x,t) = w(\frac{x-st}{\epsilon^{1/2}})$ has thickness, $\epsilon^{1/2}$.)

- Nonlinear semigroup S_t preserve certain regularizing properties of the of $S_t^\epsilon > 0$ in the limit as $\epsilon \to 0$
- Solution decays at the rate of $O(t^{-1/2})$

An Example;Riemann problem: Consider the Riemann initial data,

$$u(x,0) = \begin{cases} u_-, & \text{if } x < 0, \\ u_+, & \text{if } x > 0 \end{cases}$$

with $u_- \neq u_+$. Then the function

$$u(x,t) = \begin{cases} u_-, & \text{if } x < \dfrac{(u_- + u_+)}{2}t, \\ u_+, & \text{if } x > \dfrac{(u_+ + u_-)}{2}t \end{cases}$$

is a weak solution to inviscid Burgers equation with given data but is the one given by Hopf's formula only for the case $u_L \geq u_R$. The trouble with this solution for the case $u_l < u_R$ is that the characteristics emerge from the shock and thus violates the entropy condition. Indeed, Hopf's solution for the case $u_L < u_R$, is the continuous rarefaction solution

$$u(x,t) = \begin{cases} u_-, & \text{if } x < u_- t, \\ x/t, & \text{if} u_- t < x < u_+ t \\ u_+, & \text{if } x > u_+ t \end{cases}$$

4.2. **General Convex conservation laws -Lax's formula.** Hopf's work was extended by Lax [1957] for the convex conservation laws

$$u_t + f(u)_x = 0, x \in R^1, t > 0$$

with initial condition at $t = 0$

$$u(x,0) = u_0(x)$$

where the flux function f is strictly convex with super linear growth at infinity ie.,

$$f''(u) > 0, \frac{f(u)}{|u|} \to \infty$$

Given u_0, for fixed (x,t) introduce a family of probability measures

$$d\mu^\epsilon_{(x,t)}(y) = \frac{e^{-\frac{1}{\epsilon}\theta(x,y,t)}dy}{\int_{-\infty}^{\infty} e^{-\frac{1}{\epsilon}\theta(x,y,t)}dy},$$

where

$$\theta(x,y,t) = \int_0^y u_0(z)dz + tf*(\frac{(x-y)}{t})$$

Lax [1957] showed the following result.

Theorem: Let u_0 is bounded measurable. Then for each $t > 0$, except for a countable $x \in R^1$, $d\mu^\epsilon_{(x,t)}(y) \to \delta_{y(x,t)}$ in measure, where $y(x,t)$ is the minimizer of

$$U(x,t) = \min_{-\infty < y < \infty} \theta(x,y,t)$$

and the function $u(x,t)$ defined by

$$u(x,t) = (f^*)'(\frac{(x - y(x,t))}{t})$$

is well defined BV_{llc} function and is a weak solution of the initial value problem. Further $u(x-,t)$ and $u(x+,t)$ exists at every point and at the point of discontinuity satisfies the entropy inequality

$$u(x-,t) > u(x+,t)$$

Proof: Let us define $u^\epsilon, f^\epsilon, V^\epsilon, U^\epsilon$ by

$$\begin{aligned} u^\epsilon(x,t) &= \textstyle\int_{-\infty}^{\infty} f*'(\frac{(x-y)}{t})d\mu^\epsilon_{(x,t)} \\ f^\epsilon(x,t) &= \textstyle\int_{-\infty}^{\infty} f(f*'(\frac{(x-y)}{t}))d\mu^\epsilon_{(x,t)} \\ V^\epsilon(x,t) &= \textstyle\int_{-\infty}^{\infty} e^{\frac{-1}{\epsilon}\theta(x,t,y)}dy \\ U^\epsilon &= -\tfrac{1}{\epsilon}log(V^\epsilon) \end{aligned}$$

where we have used the identity $f(f*'(s)) = s(f*'(s)) - f*(s)$. It easy to see that $u^\epsilon = U^\epsilon_x$ and $f^\epsilon = -U^\epsilon_t$, so that $U^\epsilon_t + f^\epsilon = 0$. Taking derivative w.r.t. in this equation we get

$$u^\epsilon_t + f^\epsilon_x = 0.$$

Now take any $\phi \in C_c^\infty(R \times (0,\infty)$ multiply the equation and integrate by parts, we get

$$\int_0^\infty \int_{-\infty}^\infty (u^\epsilon \phi_t + f^\epsilon \phi_x)dxdt = 0.$$

Now apply the theorem at the appendix and a result analogous to the Lemma for the minimizer $y(x,t)$ for general convex f (which very similar to the case $f(u) = u^2/2$ done in the appendix) to get the conclusion of the theorem.

This solution satisfy the entropy condition easily follows from the increasing nature of $y(.,t)$ and $f*'$.

Finally the solution satisfies initial condition follows from the estimate

$$|U(x,t) - U(x,s)| \leq C|t-s|, \;\; |U(x,t) - \int_0^x u_0(z)dz| \leq C|t|, \;\; 0 < s < t$$

which is easy to get. Here C is a constant depends only on $f(u)$ and $||u_0||_{L^\infty}$ Details can be found in Lax (1957).

4.3. **Important Remarks :** Entropy solutions constructed above has some important structure properties we list some of them here. Also we comment on general conservation laws.

- As in the case of Hopf's solution, the present solution operator has the compactness and the regularity properties.$S_t u_0 = u(x,t)$, maps bounded subsets of L^∞ supported in a common bounded set to compact sets of L^1.
- Characteristic curves drawn in the backward direction cannot intersect any shock or any other characteristics. So every point can (x,t) can be connected by a characteristic to a point on the initial line namely $y(x,t)$.
- If additionally $f(0) = 0$ and u_0 is bounded measurable with compact support, then support of entropy solution $u(.,t)$spread at the rate $O(t^{\frac{1}{2}})$ and $sup\{|u(x,t)| : x \in R\}$ decays at the rate $O(t^{\frac{-1}{2}})$ as t goes to infinity
- If the flux is not convex there is no explicit formula for the entropy solution. One way to construct solution is by vanishing viscosity solution.

- If the flux $f(u)$ is not convex, Lax's condition is not enough to select the unique weak solution. Oleinik condition is one way to pick the unique solution: This requires
$$\frac{f(u)-f(u_-)}{u-u_-} \leq s \leq \frac{f(u)-f(u_+)}{u-u_+}$$
for all u in between u_- and u_+. This says that if $u_- > u_+$, the graph of f in (u_+, u_-) should lie below the the chord connecting $(u_+, f(u_+))$ and $(u_-, f(u_-))$ and if $u_- < u_+$, the graph of f in (u_-, u_+) should lie above the the chord connecting $(u_+, f(u_+))$ and $(u_-, f(u_-))$.
- For convex flux, Oleinik condition is equivalent to Lax's entropy in equality.
- Semi group given by weak solutions satisfying Olienik condition is L^1 contractive. This was proved by Keyfitz (1971).

5. General scaclar conservation laws in one space variable -vanishing viscosity

Here we consider general scalar conservation laws
$$u_t + f(u)_x = 0, x \in R^1, t > 0 \tag{5.1}$$
with initial condition
$$u(x,0) = u_0(x). \tag{5.2}$$
We assume $f : R^1 \to R^1$ is C^1.
Definition : A map $(\eta, q) : R^1 \to R^2$ is called an entropy-entropy flux pair for the equation if η and q are related by
$$\eta(u)'' \geq 0,, \;\; q(u)' = \eta(u)' f(u)'$$
Definition : A locally integrable function is called an entropy weak solution of the initial value problem if
$$\int_0^\infty \int_{-\infty}^\infty \eta(u(x,t))\phi(x,t)_t + q(u(x,t))\phi(x,t)_x dxdt + \int_{-\infty}^\infty \eta(u_0(x))\phi(x,0)dx \geq 0 \tag{5.3}$$
for all $\phi \in C_c(R^1 \times [0,\infty))$ with $\phi \geq 0$ and for every entropy entropy flux pair (η, q).

Remark: By taking $\eta(u) = \pm u,\; q(u) = \pm f(u)$, we see that entropy weak solution is a weak solution of the initial value problem.

Theorem : Assume u_0 is bounded measurable function which of bounded variation. Let u^ϵ is the solution of the initial value problem
$$u_t + f(u)_x = \epsilon u_{xx}, x \in R^1, t > 0 \tag{5.4}$$
with initial condition
$$u(x,0) = u_0 * e^\epsilon(x), \tag{5.5}$$
where e^ϵ is the standard Friedrichs mollifier. Then there exists smooth global solution $u^\epsilon, \epsilon > 0$ and there exists $u(x,t)$ which is pointwise limit of u^ϵ. This limit function is an entropy weak solution of (5.1) and (5.2). Further $TVu(.,t) \leq TV(u_0)$.

Proof (sketch):

- Step1. Standard fixed point argument give a C^2 solution for the parabolic approximation. Also the estimates

$$||u^\epsilon||_\infty \leq ||u_0||_\infty, \;\; \int_{-\infty}^\infty |u_x^\epsilon| dx \leq TV(u_0), \;\; \int_{-\infty}^\infty |u^\epsilon(x,t) - u^\epsilon(x,s)| dx \leq L|t-s|, t > s > 0$$

Where L depends only on $|f'(u)|$ and $||u_0||_{L^\infty}$ Apply Helly's theorem to get a subsequence $\epsilon_k$and a function u such that u^ϵ_k converge point wise a.e.

- Step 2. The limit satisfy the entropy condition.
  Multiply the PDE with $\eta(u)$ and rewrite, we get

$$\eta(u^\epsilon)_t + q(u^\epsilon)_x = \epsilon[\eta(u^\epsilon)_{xx} - \eta''(u^\epsilon)(u^\epsilon_x)^2] \leq \epsilon\eta_{xx}(u^\epsilon)$$

Now take nonnegative test function ϕ multiply integrate by parts we get,

$$\begin{aligned}&\textstyle\int_0^\infty \int_{-\infty}^\infty \eta(u^\epsilon(x,t))\phi(x,t)_t + q(u^\epsilon(x,t))\phi(x,t)_x dxdt + \int_{-\infty}^\infty \eta(u_0(x))\phi(x,0)dx \\ &\geq -\epsilon \textstyle\int_0^\infty \int_{-\infty}^\infty \eta(u^\epsilon(x,t))\phi(x,t)_{xx}.\end{aligned} \tag{5.6}$$

An application of dominated convergence theorem gives

$$\int_0^\infty \int_{-\infty}^\infty \eta(u(x,t))\phi(x,t)_t + q(u(x,t))\phi(x,t)_x dxdt + \int_{-\infty}^\infty \eta(u_0(x))\phi(x,0)dx \geq 0 \tag{5.7}$$

for all $\phi \in C_c(R^1 \times [0,\infty))$ with $\phi \geq 0$ and for every entropy entropy flux pair (η, q).

- Step 3. In the above (η, q) can be replaced by the Krushkov entropies $\eta_k(u) = |u-k|, q_k(u) = sgn(u-k)(f(u) - f(k))$. This follows by an approximation. In this case the entropy condition becomes

$$\int_0^\infty \int_{-\infty}^\infty |u-k|\phi_t + sgn(u-k)(f(u)-f(k))\phi_x dxdt + \int_{-\infty}^\infty |u_0(x)-k|\phi(x,0)dx \geq 0, \tag{5.8}$$

for all $k \in R$ and for all nonnegative test function ϕ.

- Step 4. Any weak solution which satisfies Krushkov entropy conditions is unique (see hand written notes titled Krushkov's uniqueness theorem).
- Step 5. All the sub sequential limit satisfies the Krushkov entropy conditions and hence same limit by uniqueness. So the full sequence converges and is an entropy weak solution.

5.1. **Properties of the solution operator.** Here the solution operator $S_t u_0 = u(x,t)$ means entropy weak solution with initial data u_0.

Let u_0 and v_0 are two bounded measurable functions and u and v are entropy weak solution associated with them. Let $M = sup\{|f'(u)| : u \in [inf(u_0, v_0), sup(u_0, v_0)]\}$, then

- For all $t > 0$, and every interval $[a, b]$, we have

$$\int_a^b |u(x,t) - v(x,t)|dx \leq \int_{a-Mt}^{b+Mt} |u_0(x) - v_0(x)|dx.$$

- If $u_0 - v_0 \in L^1(R^1)$, then $u(t) - v(t) \in L^1(R^1)$, $u(t) = u(.,t), v(t) = v(.,t)$ and

$$\int_{-\infty}^\infty |u(x,t) - v(x,t)|dx \leq \int_{-\infty}^\infty |u_0(x) - v_0(x)dx,$$

$$\int_{-\infty}^\infty (u(x,t) - v(x,t))dx = \int_{-\infty} \infty(u_0(x) - v_0(x))dx$$

- An easy consequence of this result is the following comparison theorem :
  If $u_0(x) \leq v_0(x)$ a.e. $x \in R^1$, then $u(x,t) \leq v(x,t)$ for a.e. (x,t)

- If $u_0 \in L^1$,then $u(t) \in L^1$ for each $t > 0$ and

$$\int_{-\infty}^{\infty} |u(x,t)|dx \leq \int_{-\infty}^{\infty} |u_0(x)|dx,$$

$$\int_{-\infty}^{\infty} u(x,t)dx = \int_{-\infty}^{\infty} u_0(x)dx$$

- If u_0 is in $BV(R^1)$, then $u(t)$ is in $BV(R^1)$ and $TV(u(.,t)) \leq TV(u_0)$.

6. Appendix

7. A useful result

Theorem 7.1. *Suppose $\theta : R^n \to R^1$, continuous growing at least quadratically at ∞. Suppose θ has a unique minimum at x_0. Then the measure Then the measure*

$$\mu_\epsilon(x) = \frac{e^{\frac{-\theta(x)}{\epsilon}}}{\int_{R^n} e^{\frac{-\theta(x)}{\epsilon}} dx}$$

converges weakly to δ_{x_0} as $\epsilon \to 0$. Further, for any continuous function that grows at most linearly at infinity

$$\lim_{\epsilon \to 0} \int_{R^n} u(x) d\mu_\epsilon(x) = u(x_0).$$

Proof. We observe that

$$\mu_\epsilon(x) = \frac{e^{\frac{\theta(x_0)-\theta(x)}{\epsilon}}}{\int_{R^n} e^{\frac{\theta(x_0)-\theta(x)}{\epsilon}} dx}$$

has the properties $\mu_\epsilon \geq 0, \int \mu_\epsilon = 1, \mu_\epsilon \to 0$, exponentially as ϵ tends to 0. So

$$\int u(x)\mu_\epsilon(x) - u(x_0) = \int (u(x) - u(x_0)) d\mu_\epsilon(x) = \int_{|x-x_0| \leq \delta} + \int_{|x-x_0| \geq \delta}$$

First ϵ tends to 0 and then let δ tends to 0. We get the result. □

Lemma: Assume that $h :\to R^1$ is a locally Lipschitz function which has at most linear growth at $\pm\infty$. For fixed $(x,t) : x \in R^1, t > 0$and let $y(x,t)$ be a minimizer for

$$min\{\frac{(x-y)^2}{2t} + h(y)\}$$

which may not be unique.

- Define $y_+(x,t) = max\{y(x,t) : y(x,t)\}$, $y_-(x,t) = min\{y(x,t) : y(x,t)\}$
- If $x_1 < x_2$, then $y(x_1,t) \leq y(x_2,t)$,
- $y_\pm$ are non decreasing functions of x , continuous except countable number of points and at the points of continuity, $y_-(x,t) = y_+(x,t)$. Further $y_-(x,t)$ is left continuous and $y_+(x,t)$ is right continuous

Proof : Because of the conditions on h, existence of minimizers $y(x,t)$, for each fixed (x,t) is clear. Let $y(x_1,t) = y_1$ and $y(x_2,t) = y_2$, then

$$Min[\frac{(x_i - y_i)^2}{2t} + h(y_i)] = \frac{(x_i - y_i)^2}{2t} + h(y_i), \ i = 1, 2.$$

We claim that

$$\{\frac{(x_2 - y_1)^2}{2t} + h(y_1)\} < \frac{(x_2 - y_)^2}{2t} + h(y_), \ \ y < y_1$$

Now take τ, $0 < \tau = \frac{y_1 - y}{x_2 - x_1 + y_1 - y} < 1$.

$$\frac{(x_2-y_1)^2}{2t} < \tau\frac{(x_1-y_1)^2}{2t} + (1-\tau)\frac{(x_2-y)^2}{2t}, \quad \frac{(x_1-y)^2}{2t} < (1-\tau)\frac{(x_1-y_1)^2}{2t} + \tau\frac{(x_2-y)^2}{2t}$$

because $x_2 - y_1 = \tau(x_1 - y_1) + (1-\tau)(x_2 - y)$ and $x_1 - y = (1-\tau)(x_1 - y_1) + \tau(x_2 - y)$

Adding these two we get

$$\frac{(x_2-y_1)^2}{2t} + \frac{(x_1-y)^2}{2t} < \frac{(x_1-y_1)^2}{2t} + \frac{(x_2-y)^2}{2t}$$

Adding $h(y_1) + h(y)$ to the above inequality and adding the resulting inequality to the very first one we get claim.

It follows that to compute the minimum we only need to consider $y \geq y_1$:

$$\min_{-\infty<y<\infty}[\int_0^y u_0(z)dz + \frac{(x_2-y)^2}{2t}] = \min_{y\geq y_1}[\int_0^y u_0(z)dz + \frac{(x-y_2)^2}{2t}] \qquad (2.6)$$

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