Transition Matrix

Consider the homogeneous system

$$\frac{dx}{dt} = A(t)x, \quad x(t_0) = x_0 \tag{1}$$

where $A(t)$ is $n \times n$ matrix whose entries are continuous functions of t and $x \in \mathbb{R}^n$. Then the system (1) has n-linearly independent solutions $x_1(t), x_2(t), \ldots, x_n(t)$. Define a nonsingular matrix $\Psi(t)$ as

$$\Psi(t) = [x_1(t), x_2(t), \ldots, x_n(t)] \tag{2}$$

Then, we have

$$\frac{\Psi(t)}{dt} = \left[\frac{dx_1(t)}{dt}, \frac{dx_2(t)}{dt}, \ldots, \frac{dx_n(t)}{dt}\right] = [A(t)x_1(t), A(t)x_2(t), \ldots, A(t)x_n(t)]$$

which gives

$$\frac{d\Psi(t)}{dt} = A(t)\Psi(t) \tag{3}$$

A nonsingular matrix $\Psi(t)$ is called a *fundamental matrix* if it satisfies the matrix differential equation (3).

A nonsingular matrix $\Phi(t, t_0)$ is called *principal fundamental matrix* or *transition matrix* or *evolution matrix* if it satisfies the matrix differential equation (3) with the initial condition $\Phi(t_0, t_0) = I$, which is given by

$$\Phi(t, t_0) = \Psi(t)\Psi^{-1}(t_0).$$

Write MATLAB code to solve the following problems.

1. For the following matrices, find e^{At}

(a) $A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, (b) $A_2 = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$, where a and b are scalars,

(c) $A_3 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix}$, (d) $A_4 = \begin{pmatrix} -7 & -9 & 9 \\ -3 & 5 & -3 \\ -3 & -3 & 5 \end{pmatrix}$.

2. Verify that $\Psi(t) = \begin{pmatrix} e^{-2t}\cos t & -\sin t \\ e^{-2t}\sin t & \cos t \end{pmatrix}$ is a fundamental matrix of the system

$$\frac{dx(t)}{dt} = \begin{pmatrix} -2\cos^2 t & -1-\sin 2t \\ 1-\sin 2t & -2\sin t \end{pmatrix} x(t).$$

3. For $A = \begin{pmatrix} -7 & -9 & 9 \\ -3 & 5 & -3 \\ -3 & -3 & 5 \end{pmatrix}$, solve the following homogeneous initial value problem

$$\frac{dx(t)}{dt} = Ax(t), \quad x(0) = (1,1,1)^T.$$

4. Solve the initial value problem

$$\frac{dx(t)}{dt} = Ax(t), \quad x(0) = (1,t)^T$$

where **(a)** $A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 2 & 0 & 1 & 0 \end{pmatrix}$ and **(b)** $A = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$.

5. Use transition matrix to solve the following system of equation

$$\frac{dx(t)}{dt} = Ax(t) + b(t), \quad x(t_0) = x_0$$

where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $b(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}$.

6. The oscillations of a particle of mass m is given by the following second order differential equation

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \omega^2 x = F(t), \quad x(t_0) = k_1 \text{ and } \dot{x}(t_0) = k_2$$

where k is resistance and $\omega^2 = \frac{\lambda}{ma}$. Use transition matrix to find the solution.

7. Use transition matrix to find the solution of the following system of equations

$$\frac{dx(t)}{dt} = Ax(t) + b(t), \quad x(t_0) = x_0$$

where $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$.

****** ***THE END*** ******

Controllability

Consider the linear system

$$\begin{aligned} \frac{dx}{dt} &= A(t)x(t) + B(t)u(t) \\ x(t_0) &= x_0 \end{aligned} \tag{1}$$

where $A(t)$ is $n \times n$ matrix, $B(t)$ is $n \times m$ matrix, $u(t) \in \mathbb{R}^m$ and $x(t) \in \mathbb{R}^n$. $u(t)$ is called control or input vector and $x(t)$ is the corresponding trajectory or state of the system.

The typical controllability problem involves the determination of the control vector $u(t)$ such that the state vector $x(t)$ has the desired properties. Also, we assume that the entries of the matrices $A(t)$ and $B(t)$ are continuous so that the above system has a unique solution $x(t)$ for a given input $u(t)$.

The linear system (1) is said to be *controllable* if given any initial state x_0 and any final state x_f in $\mathbb{R}^n$, there exist a control $u(t)$ so that the corresponding trajectory $x(t)$ of (1) satisfies the condition

$$x(t_0) = x_0 \text{ and } x(t_f) = x_f.$$

Write MATLAB code to find the solution of the following problems:

1. Consider the linear system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

and $x_0 = (0,0)^T$ and $x_f = (1/2, 1/2)^T$. Find the control $u(t)$ which steers the system from initial state x_0 to the final state x_f.

2. Discuss the controllability of the following input-output system

$$\frac{dx}{dt} = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t).$$

3. Discuss the controllability of the following system

$$\frac{dx(t)}{dt} = Ax(t) + b(t), \quad x(t_0) = x_0$$

where $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

4. Consider the system

$$\frac{dx(t)}{dt} = Ax(t) + b(t), \quad x(t_0) = x_0$$

where $A = \begin{pmatrix} -2 & 0 & 1 \\ 0 & -1 & 0 \\ -3 & 4 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$. Determine whether the given system is controllable or not? If yes, find the control $u(t)$ which steers the system from initial state $(0,0,0)^T$ to the final state $(1,1,1)^T$ at time $T_f = 1$.

5. Consider the satellite problem

$$\frac{dx(t)}{dt} = Ax(t) + b(t), \quad x(t_0) = x_0$$

where $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$, $x \in \mathbb{R}^4$ and $u \in \mathbb{R}^2$. Show that the above system is controllable and find a steering controller which steers the system from initial state $x_0 = (1,2,3,4)^T$ to the final state $x_f = (4,3,2,1)^T$ at time $T_f = 1$.

****** ***THE END*** ******