AN INTRODUCTION TO THE THEORY OF VISCOSITY SOLUTIONS

K.S. MALLIKARJUNA RAO

1. Introduction

1.1. **Necessity of a nonsmooth solution.** We start with few difficulties that arise in the study partial differetial equations.

Consider the following simple equation

$$\begin{cases} |u'| = 1, & x \in (0,1) \\ u(0) = u(1) = 0. \end{cases} \tag{1.1}$$

The first question is the existece of a classical solution. By a classical solution, we mean a contiuously differentiable function $u : [0,1] \to \mathbb{R}$ which satisfies (1.1). Since $|u'|$ is one, if u is classical solution, u' has to be identically equal to either 1 or -1. Thus one of the boundary condition will fail. Thus a classical solution does not exist.

Due to the nonlinearity of the equation, we can not apply the weak solution techniques. The next option is to try out the generalized solutions. A generalized solution is a function u which satisfies the equation in a.e. sense.

Define the function $u_n, n = 1, 2, \cdots$ as follows:

$$\begin{aligned} u_n(0) &= 0, \\ u_n'(x) &= \begin{cases} 1, \text{ if } x \in (\frac{2k}{2^n}, \frac{2k+1}{2^n}) \\ -1, \text{ if } x \in (\frac{2k+1}{2^n}, \frac{2k+2}{2^n}) \end{cases} \end{aligned}$$

for $k = 0, 1, \cdots, 2^{n-1} - 1$. Then it is easy to see that u_n's are infinitely many solutions of the equation (1.1). This leads us to the problem of nonuniqueness. There is another problem associated to this example. Note that the solutions u_n converge to zero as $n \to \infty$ and zero is not a solution. Thus the approximations does not give approximate solutions. This is called stability. Stability is an important issue from the physics point of view.

In order to get rid of these troubles, Crandall & Lions introduced the notion of viscosity solutions to nonlinear pde. There are some other notions of solutions like Clarke's generalized solution and Subbotin's minimax solution. However, they are specific to the first order equations with special dependence on gradient variable.

1.2. **Viscosity solutions: Definition and Stability.** Consider the general nonlinear equation

$$F(x, u(x), Du(x), D^2u(x)) = 0, x \in \Omega \tag{1.2}$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathscr{S}^N \to \mathbb{R}$ is a continuous function and Ω is any open set in $\mathbb{R}^N$. Here $\mathscr{S}^N$ denotes the space of all $N \times N$ real valued symmetric matrices.

We call F degenerate elliptic if the following condition is satisfied:

$$F(x, r, p, X) \leq F(x, r, p, Y) \text{ whenever } X \geq Y$$

where $X \geq Y$ means that $X - Y$ is nonnegative definite matrix. It is called proper if it is degenerate elliptic and satisfies the monotonicity condition in the r-variable i.e.,

$$F(x, r, p, X) \leq F(x, s, p, Y) \text{ whenever } r \leq s \text{ and } X \geq Y.$$

Date: June 12, 2014.

From now on, we always assume our equations to be proper, unless otherwise specified. To motive the definition of viscosity solutions, we prove the following proposition which is a variant of classical maximum principle.

Proposition 1.1. *Let u be a classical solution of the equation (1.2) and $\phi : \Omega \to \mathbb{R}$ be any C^2 function. Then if $u - \phi$ has a locam maximum (local minimum) at a point $x_0 \in \Omega$, then*

$$F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \leq (\geq) 0.$$

Proof. The proof is a simple consequence of the maximum principle. If $u-\phi$ has a local maximum at x_0, then $Du(x_0) = D\phi(x_0)$ and $D^2u(x_0) - D^2\phi(x_0) \leq 0$. Now using the degenerate ellipticity, we conclude. □

We now give the definition of the viscosity solution.

Definition 1.2. *An upper semicontinuous function (resp., lower semincontinuous function) $u : \Omega \to \mathbb{R}$ is called a viscosity subsolution (resp., viscosity supersolution) if*

$$F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \leq (\text{resp., } \geq) 0$$

whenever x_0 is local maximum (resp., local minimum) of $u - \phi$ for a smooth function $\phi : \Omega \to \mathbb{R}$.

A continuous function u which is both viscosity sub and super-solution is called a viscosity solution.

Few remarks are in order.

Remark 1.3. *The definition of viscosity solution is a local one. This means that u is viscosity subsolution in Ω then it is subsolution also in Ω' where $\Omega' \subset \Omega$.*

Remark 1.4. *In the defintion of viscosity solution, local maximum can be replaced by global maximum and also by strict local or global maximum. Also C^2 functions can be replaces by smooth functions. Also we can assume that the local maximum is zero. Similar remark applies for supersolutions also.*

Now on, we remove the term viscosity and we simply call subsolution or supersolution unless no confusio arises. We also use the following notation throughout: the function ϕ used in the definition of viscosity solutions are called test functions.

We now consider the Dirichlet problem associated with (1.2). Let $u_0 : \partial\Omega \to \mathbb{R}$ be a given function.

Definition 1.5. *An usc (resp. lsc) function u is called subsolution (resp. supersolution) of (1.2) with the boundary condition*

$$u(x) = u_0(x), \text{ on } \partial\Omega \tag{1.3}$$

if u is subsolution (resp., supersolution) in Ω and satisfies

$$u(x) \leq (\geq) u_0(x) \text{ on } \partial\Omega.$$

A continuous function u is called solution to the Dirichlet problem (1.2) - (1.3) if it is both sub and super solution.

We now present the corner stone of the theory of viscosity solutions i.e., the stability result.

Theorem 1.6. (Stability) *Let $F, F_n, n = 1, 2, \cdots$ be proper and assume that $F_n \to F$ uniformly on compact sets. Let u_n be viscosity solution to the equation*

$$F_n(x, u_n(x), Du_n(x), D^2u_n(x)) = 0 \text{ in } \Omega$$

and assume that $u_n \to u$ uniformly on compact sets. Then u is viscosity soltion to (1.2).

Proof. We prove the subsolution part and leave the supersolution part as it is similar.

Let $u - \phi$ has a strict local maximum at $x_0 \in \Omega$ where ϕ is a smooth function. Let B be a ball around x_0 such that

$$u(x_0) - \phi(x_0) = \sup_B (u - \phi) > \sup_{\partial B} (u - \phi).$$

Choose x_n such that

$$u_n(x_n) - \phi(x_n) = \sup_B (u_n - \phi).$$

Choose a subsequence, which we again denote by the same by an abuse of notation, such that $x_n \to \bar{x}$ for some $\bar{x} \in \bar{B}$. Now for any $x \in B$, we have the following:

$$u_n(x) - \phi(x) \leq u_n(x_n) - \phi(x_n) \to u(\bar{x}) - \phi(\bar{x}) \text{ as } n \to \infty.$$

Thus

$$u(x) - \phi(x) \leq u(\bar{x}) - \phi(\bar{x})$$

for all $x \in B$ and hence $\bar{x} = x_0$ due the strictness of local maximum. Now using the subsolution property for u_n, we have

$$F(x_n, u(x_n, D\phi(x_n), D^2\phi(x_n)) \leq 0.$$

Now letting $n \to \infty$, we obtain

$$F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \leq 0.$$

This completes the theorem. □

We now look at the example considered in previous section.

Example 1.7. *Consider the exampe considered in Section 1. Define*

$$u(x) = \frac{1}{2} - |\frac{1}{2} - x|$$

We now verify that u is viscosity solution of (1.1). If ϕ is any test function such that $u - \phi$ has a local extremum at any point other that $\frac{1}{2}$, then $u'(x) = \phi'(x) = \pm 1$ and hence u satisfies both the subsolution and supersolution properties. If $\frac{1}{2}$ is a local maximum of $u - \phi$, then using simple calculus, we note that $\phi'(\frac{1}{2}) \in [-1, 1]$. Thus u satisfies the subsolution property at $\frac{1}{2}$. Now let $u - \phi$ has local minimum at $\frac{1}{2}$, then observe that $1 \leq \phi'(\frac{1}{2}) \leq -1$ which is not possible. Thus $u - \phi$ can not have any local minimum for any smooth function. Thus u is viscosity solution.

We now prove that any other generalized solution can not be viscosity solution. Since any other generalized solution have a local mimum and at that local minimizer the supersolution property fails to hold, this follows.

1.3. **Sub and Super differentials.** In this section, we give an alternate definition of viscosity solution which will be useful in certain cases.

Definition 1.8. *Let $u : \Omega \to \mathbb{R}$. Define the sets $D^+u(x), D^-u(x), J^{2,+}u(x), J^{2,-}u(x)$ which are respectively called super differential, sub differential, superjet and subject as follows:*

$$D^+u(x) = \{p \in \mathbb{R}^N : \limsup_{y \to x} \frac{u(y) - u(x) - \langle p, y - x\rangle}{|y - x|} \leq 0\}$$

$$D^-u(x) = \{p \in \mathbb{R}^N : \liminf_{y \to x} \frac{u(y) - u(x) - \langle p, y - x\rangle}{|y - x|} \geq 0\}$$

Remark 1.9. *If $(p, X) \in J^{2,+}u(x)$, it is obvious from the definition that $(p, Y) \in J^{2,+}u(x)$ for all $Y \geq X$. Thus $J^{2,+}u(x)$ is always either empty or infinite. Also the superjet need not be closed if it is nonempty. Similar remark holds for the superjet. However, sub and superdifferentials can be finite sets and are always closed.*

We now prove a characterization of these sets which gives an equivalent definition of viscosity solution in terms of these sets.

Proposition 1.10.

$$D^+u(x) = \{D\phi(x) : u - \phi \text{ has a local maximum at } x \text{ for some test function } \phi\}$$
$$D^-u(x) = \{D\phi(x) : u - \phi \text{ has a local minimum at } x \text{ for some test function } \phi\}$$

Proof. We will only prove the third part and rest can be proved similarly or using the dual relation between the sub and super jets.

If ϕ is a test function such that $u-\phi$ has local maximum at x, then using the second order taylor's series expansion, we can easily show that

$$(D\phi(x), D^2\phi(x)) \in J^{2,+}u(x).$$

We now show the converse.

Let $(p, X) \in J^{2,+}u(x)$. By definition we can choose a continuous, non-decreasing function σ such that

$$u(y) = u(x) + \langle p, y-x\rangle + \frac{1}{2}\langle X(y-x), y-x\rangle + \sigma(|y-x|)|y-x|^2$$

Now define a C^2 function η by

$$\eta(r) = \int_0^r \int_0^s \sigma(\tau)d\tau \; ds$$

Then we have

$$\eta(4r) \geq 2r^2\sigma(r).$$

Let

$$\phi(y) = \phi(x) + \langle p, y-x\rangle + \frac{1}{2}\langle X(y-x), y-x\rangle + \eta(4|y-x|)$$

then η is C^2 funtion. Note that

$$u(y)-\phi(y) \leq \sigma(|y-x|)|y-x|^2 - \eta(4r) \leq \sigma(|y-x|)|y-x|^2 - 2\sigma(|y-x|)|y-x|^2 \leq 0 = u(x)-\phi(x)$$

and thus $u-\phi$ has local maximum at x. Also note that $(p, X) = (D\phi(x), D^2\phi(x))$. This proves the reverse inclusion. ☐

This proposition gives a new definition to the viscosity solutions.

Definition 1.11. *An usc function u is said to be viscosity subsolution if*

$$F(x, u(x), p) \leq 0 \text{ for all } p \in D^+u(x)$$

and a lsc function u is said to be viscosity supersolution if

$$F(x, u(x), p, X) \geq 0 \text{ for all } p \in D^-u(x).$$

A viscosity solution is both sub and supersolution.

1.4. **Some Properties of Sub and Super differentials.** In this section, we review some properties of sub and super differentials.

Theorem 1.12. 1. *$D^+u(x)$ and $D^-u(x)$ are closed and convex.*

2. *If both $D^+u(x)$ and $D^-u(x)$ are non empty, then u is differentiable at x and $D^+u(x) = D^-u(x)$, a singleton.*
3. *$\{x \in \Omega : D^+u(x) \neq \emptyset\}$ and $\{x \in \Omega : D^-u(x) \neq \emptyset\}$ are dense in Ω.*

Proof. We will prove only (2) and (3) as (1) is trivial.

Since both $D^+u(x), D^-u(x)$ are nonempty, we can find test functions $\phi_1(x), \phi_2(x)$ such that

$$\phi_1(y) \leq u(y) \leq \phi_2(y) \text{ and } \phi_1(x) = u(x) = \phi_2(x)$$

Now using classical maximum principle, we obtain that $D\phi_1(x) = D\phi_2(x)$. Thus $D^+u(x) = D^-u(x)$, a singleton. Now by the definition of sub and superdifferentials, we note that

$$\lim_{y\to x} \frac{u(y)-u(x)-\langle p, y-x\rangle}{|y-x|} = 0$$

where $p \in D^+u(x)$. Thus u is differentiable at x. This completes part (2).

Let $\bar{x} \in \Omega$ and B be an open ball centered at $\bar{x}$ in Ω. Let $\epsilon > 0$ and consider the test function $\phi(x) = \frac{1}{\epsilon}|x-\bar{x}|$. Choose x_ϵ such that

$$u(x_\epsilon) - \phi(x_\epsilon) = \sup_{x\in B}(u-\phi)$$

Now

$$u(\bar{x}) \leq u(x_\epsilon) - \phi(x_\epsilon)$$

and hence

$$\frac{1}{\epsilon}|x_\epsilon - \bar{x}| \le C$$

where $C = 2\sup_B u$. Hence $x_\epsilon \to \bar{x}$. Therefore $x_\epsilon \in B$ for all ϵ sufficiently small. Hence $D^+u(x_\epsilon) \neq \emptyset$ for all such ϵ. Similarly we can prove the subdifferential part. This completes the proof of part (3) and hence the theorem. □

We now state the dual relationship between sub and superdifferentials. We omit the proof as it is trivial.

Theorem 1.13. *For any usc function $u : \Omega \to \mathbb{R}$, we have*

$$D^+u(x) = -D^-(-u)(x) \text{ and } J^{2,+}u(x) = -J^{2,-}(-u)(x)$$

2. Some Properties of Viscosity Solutions

In this section, we present some properties of viscosity solutions.

Proposition 2.1. (a) *Let $u, v \in C(\Omega)$ be viscosity subsolutions of (1.2). Then $u \vee v$ is also a subsolution of (1.2).*

(b) *Let $u, v \in C(\Omega)$ be viscosity supersolutions of (1.2). Then $u \wedge v$ is also a supersolution of (1.2).*

(c) *Let $u \in C(\Omega)$ be a viscosity subsolution of (1.2) such that $u \ge v$ for any subsolution $v \in C(\Omega)$. Then u is a viscosity solution of (1.2).*

Proof. Let $u \vee v - \phi$ has a local maximum at x_0. With out loss of generality, we can assume that $u(x_0) = u \vee v(x_0)$. It is clear to see that $u - \phi$ also has local maximum at x_0. Thus

$$F(x_0, u \vee v(x_0), D\phi(x_0), D^2\phi(x_0)) = F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \le 0$$

Thus $u \vee v$ is a subsolution. This completes the proof of (a). Similarly we can prove (b).

We now proceed to prove (c). The proof is by contradiction. Assume that u is not viscosity supersolution. Thus there is a point x_0 and a smooth function ϕ such that $u - \phi$ has local minimum at x_0 and

$$\theta = F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) < 0$$

Choose $\delta_0 > 0$ such that

$$u(x_0) - \phi(x_0) \le u(x) - \phi(x) \text{ for all } x \in \bar{B}(x_0, \delta_0)$$

Consider the smooth function

$$\psi(x) = \phi(x) - |x - x_0|^4 + u(x_0) - \phi(x_0) + \frac{1}{2}\delta^4$$

for $0 < \delta < \delta_0$. Note that

$$u(x_0) - \psi(x_0) < u(x) - \psi(x)$$

whenever $|x - x_0| = \delta$. Note that

$$\psi(x_0) = u(x_0) + \frac{1}{2}\delta^2, D\phi(x_0) = D\psi(x_0) \text{ and } D^2\phi(x_0) = D^2\psi(x_0).$$

Hence we can choose δ sufficiently small so that

$$|F(x, \psi(x), D\psi(x), D^2\psi(x)) - F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0))| + \theta \le 0$$

for all $x \in \bar{B}(x_0, \delta)$. Now define $\bar{v}$ by

$$\bar{v} = \begin{cases} u \vee \psi \text{ on } B(x_0, \delta) \\ u \text{ on } \Omega \setminus B(x_0, \delta) \end{cases}$$

Then it is easy to see that $\bar{v}$ is a continuous function and is subsolution. Since $\bar{v}(x_0) > u(x_0)$, this gives contradiction to our hypothesis. Thus u is viscosity solution. This completes the proof of part (c) and hence the theorem. □

We now prove change of unknown.

Proposition 2.2. *Let $u \in C(\Omega)$ be a viscosity solution and $\Phi \in C^2(\mathbb{R})$ such that*

2.1. **Vanishing Viscosity Method.** In this section, we introduce the vanishing viscosity method which is the basis for the name "viscosity solution". This also gives an application to the stability result of viscosity solution.

Consder the Hamilton-Jacobi equation

$$u + H(Du) = f, \ x \in \mathbb{R}^N \tag{2.1}$$

where $H : \mathbb{R}^N \to \mathbb{R}$ is a Lipschitz continuous function. We assume that $f : \mathbb{R}^N \to \mathbb{R}$ is bounded and Lipschitz continuous function.

Consider the second order equation

$$-\epsilon\Delta u_\epsilon + u_\epsilon + H(Du_\epsilon) = f. \tag{2.2}$$

Due to the uniform ellipticity of (2.2), there is a unique classical solution of (2.2) with the property

$$\|u_\epsilon\|_\infty \leq \|f\|_\infty \tag{2.3}$$

Further if v_ϵ denotes the classical solution of (2.2) where f is replaced by any other bounded Lipschitz continuous function g, we have from the maximum principle, the following estimate

$$\|u_\epsilon - v_\epsilon\|_\infty \leq \|f - g\|_\infty \tag{2.4}$$

Now for any fixed $h \in \mathbb{R}^N$, let g be defined by $g(x) = f(x + h)$. Then $v_\epsilon(x) = u_\epsilon(x + h)$ for all $x \in \mathbb{R}^N$. Now using (2.3) and (2.4), we obtain the following estimates

$$\|u_\epsilon\|_\infty \leq \|f\|_\infty \text{ and } \|u_\epsilon(\cdot) - u_\epsilon(\cdot + h)\|_\infty \leq C|h|$$

where C is the Lipschitz constant of f. Using the above inequalities, we conclude that $\{u_\epsilon\}$ is equibounded and equiLipschitz continuous. Now applying Ascoli-Arzela's theorem, $u_\epsilon \to u$ locally uniformly along a subsequenece, which we denote again by u_ϵ by an abuse of notation. The stability of viscosity solutions yields that u is a viscosity solution of (2.1).

2.2. **Exercises.**

(1) Show by a density argument that an equivalent definition of viscosity solution can by given by using smooth test functions instead of C^2 test functions.
(2) Show that if $D^+u(x) \neq \emptyset$ (or $J^{2,+}u(x) \neq \emptyset$) then u is usc at x. Similarly if $D^-u(x)$ (or $J^{2,-}u(x)$) is nonempty then u is lsc. (Remark: This exercise justifies the reason for taking the semicontinuity in the definition of viscosity solutions).
(3) If u is any Lipschitz continuous, then show that $D^+u(x)$ and $D^-u(x)$ are contained in the ball $\bar{B}(0, L)$.
(4) Let $u : \mathbb{R}^N \to \mathbb{R}$ be convex. The subdifferential of u in the sense of convex analysis is the set
$$\partial u(x) = \{p \in \mathbb{R}^N : u(y) \geq u(x) + p \cdot (y - x), \forall y \in \mathbb{R}^N\}$$
Show that $\partial u(x) = D^-u(x)$.
(5) Assume that $u_n \in C(\Omega$ and $u_n \to u$ locally uniformly in Ω. Show that for any $x \in \Omega$,
$$D^+u(x) \subseteq \limsup_{\substack{n\to\infty \\ y\to x}} D^+u_n(y).$$
(6) Let $u \in C([a, b])$. Prove the mean value property: there exists $\xi \in (a, b)$ such that $u(b) - u(a) = p(b - a)$ for some $p \in D^-u(\xi) \cup D^+u(\xi)$.

3. Comparision Principles and Uniqueness

In this chapter, we devote our attention to study comparison principles and uniqueness results. We start with first order equations

3.1. **First Order Equations.**

3.2. **Uniquness in Bounded Domains.** Consider the fully nonlinear first order equation

$$H(x, u(x), Du(x)) = 0, x \in \Omega \tag{3.1}$$

where Ω is a bounded open set in $\mathbb{R}^N$. We use the symbol H in first order case to signify that it is Hamiltonian in most of the examples. We always assume that H is strictly monotonically increasing in the u-variable. Formally this amounts to the following condition

$$\frac{\partial H(x, u, p)}{\partial u} > \gamma > 0.$$

If H is not differentiable, then the above assumption reads as
Assumption (H1)

$$H(x, u, p) - H(x, v, p) \geq \gamma(u - v)$$

for all $u \geq v$.

This guarantees uniqueness as in the case of implicit function theorem. Suppose if this condition is not true. Then one can easily see that any translation of a solution also becomes a solution. Thus this condition is essential in order to have uniqueness. We will also study the equations which do not depend on u-variable. In that case, the uniqueness criteria has to be modified.

Assume u and v are classical sub and super solutions respectively of (3.1). Also assume that

$$u \leq v \text{ on } \partial\Omega.$$

We want to show that $u \leq v$ in Ω. Suppost $u \not\leq v$. Then there is a point $x_0 \in \Omega$ such that $u - v$ has maximum at x_0. Now using classical maximum principle we note that $Du(x_o) = Dv(x_0)$. Thus we have

$$H(x_0, u(x_0), Du(x_0)) - H(x_0, v(x_0), Dv(x_0)) \leq 0$$

which gives a contradiction to the assumption (H1). Thus $u \leq v$. In the case of viscosity solutions, this proof does not work directly as $Du(x_0)$ may not be defined. However, using Theorem 1.4.1 (3) can be used to overcome this difficulty. We now proceed to prove the comparison principles for viscosity solutions.

Let u, v be sub and supersolution of (3.1). Assume that $u \leq v$ on $\partial\Omega$. Consider the the function

$$\Phi(x, y) = u(x) - v(y) - \frac{1}{\epsilon}|x - y|^2, \; x, y \in \bar{\Omega}$$

where $\epsilon > 0$ ia a parameter. Due to the semincontinuity of u, v and boundedness of the domain, $u, -v$ are bounded above. Thus Φ is bounded above for each $\epsilon > 0$. Let (x_ϵ, y_ϵ) be a maximizer of Φ.

As earlier assume that $u \not\leq v$. Thus there exists $x_0 \in \Omega$ such that

$$u(x_0) - v(x_0) = \sup_{\bar{\Omega}}(u - v) = \delta_0 > 0.$$

Clearly

$$\delta_0 \leq \Phi(x_\epsilon, y_\epsilon).$$

Thus

$$\frac{1}{\epsilon}|x_\epsilon - y_\epsilon|^2 \leq u(x_\epsilon) - u(y_\epsilon) - \delta_0.$$

Since RHS is bounded, $|x_\epsilon - y_\epsilon| \to 0$. Thus $x_\epsilon, y_\epsilon \to \bar{x}$ for some $\bar{x} \in \Omega$. Using this convergence again in the above inequality we get that

$$\limsup_{\epsilon \to 0} \frac{1}{\epsilon}|x_\epsilon - y_\epsilon|^2 \leq 0$$

and hence

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}|x_\epsilon - y_\epsilon|^2 = 0.$$

Note that $x \mapsto u(x) - v(y_\epsilon) - \frac{1}{2}|x - y_\epsilon|^2$ has maximum at x_ϵ and $y \mapsto v(y) - u(x_\epsilon) + \frac{1}{2}|x_\epsilon - y|^2$ has minimum at y_ϵ. By definition of viscosity solutions, we have

$$H(x_\epsilon, u(x_\epsilon), \frac{2}{\epsilon}(x_\epsilon - y_\epsilon)) \leq 0$$
$$H(y_\epsilon, u(y_\epsilon), \frac{2}{\epsilon}(x_\epsilon - y_\epsilon)) \geq 0$$

Subtracting, we get

$$H(x_\epsilon, u(x_\epsilon), \frac{2}{\epsilon}(x_\epsilon - y_\epsilon)) - H(y_\epsilon, u(y_\epsilon), \frac{2}{\epsilon}(x_\epsilon - y_\epsilon)) \leq 0 \tag{3.2}$$

Suppose H is given by

$$H(x, u, p) = u + \bar{H}(p) - f(x) \tag{3.3}$$

then (3.2) transforms into

$$u(x_\epsilon) - v(y_\epsilon) + f(y_\epsilon) - f(x_\epsilon) \leq 0.$$

Letting $\epsilon \to 0$, we obtain $\delta_0 \leq 0$ which a contradiction to our assumption. Thus we have proved the following theorem.

Theorem 3.1. *Let u, v be sub and supersolutions of (3.1) where H is given by (3.3). Assume $\bar{H}, f$ are continuous. Let $u \leq v$ on $\partial\Omega$. Then*

$$u \leq v \ \textit{in}\ \Omega.$$

We now proceed to prove the general case. We need the followig condition:
Assumption (H2)

$$H(x, u, \alpha(x-y)) - H(x, u, \alpha(x-y) \leq \omega(|x-y|(1+\alpha|x-y|))$$

where ω is a modulus of continuity i.e., nonegative continuous and increasing function with the property $\omega(r) \to 0$ as $r \downarrow 0$.

Theorem 3.2. *Let u, v be sub and supersolutions of (3.1). Assume (H1) and (H2). Let $u \leq v$ on $\partial\Omega$. Then*

$$u \leq v \ \textit{in}\ \Omega.$$

Proof. Using the assumption (H2) in (3.2) we obtain

$$H(x_\epsilon, u(x_\epsilon), \frac{2}{\epsilon}(x_\epsilon - y_\epsilon)) - H(x_\epsilon, u(y_\epsilon), \frac{2}{\epsilon}(x_\epsilon - y_\epsilon)) - \omega(|x_\epsilon - y_\epsilon|(1 + \frac{2}{\epsilon}|x_\epsilon - y_\epsilon|)) \leq 0$$

Now using the assumption (H1), we obtain

$$\gamma(u(x_\epsilon) - v(y_\epsilon)) - \omega(|x_\epsilon - y_\epsilon|(1 + \frac{2}{\epsilon}|x_\epsilon - y_\epsilon|)) \leq 0$$

Letting $\epsilon \to 0$, we obtain $\gamma\delta_0 \leq 0$, which is a contradiction. Thus $u \leq v$ in Ω. ☐

As a corollary we obtain the following uniqueness result for the viscosity solutions of (3.1).

Corollary 3.3. *Let u, v be two viscosity solutions of (3.1) such that $u = v$ on $\partial\Omega$. Then $u = v$. Thus the Dirichlet problem*

$$\begin{cases} H(x, u, Du(x)) = 0 \ \textit{in}\ \Omega \\ u = u_0 \ \textit{on}\ \partial\Omega \end{cases}$$

has at most one solution.

3.3. **Uniqueness in Unbounded Domains.** Here we study the first order equation (3.1) where Ω is unbounded. We have used the fact that Ω is bounded very crucially in the previous subsection. Due to the boundedness of Ω, the solutions are bounded. This fact is used very crucially in the proof of comparison principle. If we make the boundedness of the solutions as an assumption, we still able to conclude the comparison principle in the unboundedness case. However the function Φ has to be modified suitably. We now prove the comparison principle for the bounded viscosity solutions of (3.1). Due to the lack of compactness of the domain, we need the following condition on H:

Assumption (H3)

$$H(y,u,\lambda(x-y)+p) - H(x,u,\lambda(x-y)+q) \leq \omega(|x-y| + \lambda|x-y|^2) + \bar{\omega}(|p-q|)$$

for all $\lambda \geq 1, p, q \in \bar{B}(0,1)$ and $x, y \in \Omega$.

Theorem 3.4. *Let u, v be bounded sub and supesolutions of (3.1) respectively. Assume that $u \leq v$ on $\partial\Omega$. Then $u \leq v$.*

Proof. Consider the function

$$\Phi(x,y) = u(x) - v(y) - \frac{1}{\epsilon} - \alpha(\eta(x) + \eta(y))$$

where $\epsilon, \alpha > 0$ and η is any nonnegative Lipschitz continuous and smooth function with the following property

$$\eta(x) \to \infty \text{ as } |x| \to \infty.$$

As we are dealing with second order equations, η should be twice continuous differentiable at least. An example of such a function is $\eta(x) = \sqrt{(1+|x|^2)}$. The reason for adding β to Φ is the following fact: If we do not add this, then we will not be able to choose a maximizer of Φ in $\Omega \times \Omega$.

Now let $(\bar{x}, \bar{y}) \in \Omega \times \Omega$ be a maximizer of Φ in $\mathbb{R}^N \times \mathbb{R}^N$.

As in previous case, we proceed by contradiction. Let $x_0 \in \Omega$ be such that

$$\delta_0 = u(x_0) - v(x_0) > 0.$$

Thus we have the following inequalities

$$\begin{aligned} &\delta_0 - 2\alpha\eta(x_0) \leq \Phi(\bar{x}, \bar{y}), \\ &\tfrac{1}{\epsilon}|\bar{x} - \bar{y}|^2 \leq u(\bar{x}) - v(\bar{y}). \end{aligned}$$

As earlier, we obtain from the second inequality that

$$|\bar{x} - \bar{y}| \to 0 \text{ and } \frac{1}{\epsilon}|\bar{x} - \bar{y}|^2 \to 0$$

as $\epsilon \to 0$.

Now using the definition of viscosity solution, we obtain the following inequalities:

$$H(\bar{x}, u(\bar{x}), \frac{2}{\epsilon}(\bar{x} - \bar{y}) + \alpha D\eta(\bar{x})) \leq 0$$
$$H(\bar{y}, u(\bar{y}), \frac{2}{\epsilon}(\bar{x} - \bar{y}) - \alpha D\eta(\bar{y})) \leq 0$$

If we choose α sufficiently small such that $\alpha\|D\eta\|_\infty < 1$, we can apply the assumption (H3). Now applying (H1) and (H3), we obtain

$$\gamma(u(\bar{x} - v(\bar{x}) - \omega(|\bar{x} - \bar{y}| + \frac{2}{\epsilon}|\bar{x} - \bar{y}|^2) - \bar{\omega}(\alpha|D\eta(\bar{x} + D\eta(\bar{y})|)$$

Thus we have

$$\delta_0 - 2\alpha\eta(x_0) \leq \omega(|\bar{x} - \bar{y}| + \frac{2}{\epsilon}|\bar{x} - \bar{y}|^2) + \bar{\omega}(\alpha|D\eta(\bar{x} + D\eta(\bar{y})|)$$

Now letting $\epsilon \to 0$ first and then letting $\alpha \to 0$, we arrive at contradiction. Thus the proof is completed. ☐

In the above theorem, we assumed the boundedness of solutions. But in many cases, the boundedness seems to be very restrictive assumptions. In the following thoerem, we relax this and instead, we assume the uniform continuity.

Theorem 3.5. *Let u, v be uniformly continuous sub and supersolutions of (3.1) respectively. Assume that $u \leq v$ on $\partial\Omega$. Then $u \leq v$.*

Proof. The proof is based on the following observation. If u is uniformly continuous with the modulus of continuity ω, then ω satisfies

$$\sup_{r>0} \frac{\omega(r)}{1+r} < \infty.$$

Once we have this fact, the rest of the proof is similar. Due to this fact, the function Φ defined earlier will have maximum attained by a point. We donot give the details of the proof as it is by now routine. □

Remark 3.6. *So far, all our assumptions are global. However, this is not necessary. We can assume that (H2) and (H3) are satisfied locally. By a slight modification of the proofs, we can still prove the comparison result and hence uniquencess of Dirichlet problem.*

3.4. **Evolution Equations and Cauchy Problem.** In this section, we concentrate on the evolution equations. We consider the following equation

$$\frac{\partial u}{\partial t} + H(t, x, u, Du) = 0, (t, x) \in (0, T] \times \mathbb{R}^N. \tag{3.4}$$

We have the following theorem.

Theorem 3.7. *Let H be continuous and satisfy (H3). Let u, v be uniformly contiuous and viscosity subsolution and viscosity supersolution respectively of (3.4). Assume that $u(0, x) \leq v(0, x)$. Then $u \leq v$.*

4. Existence: Perron's Method

5. Discontinuous Solutions and Barles-Perthame's Method

One of the most important result in the theory of visosity solution is the stability result. Roughly it says that if the solutions of a family of PDEs converge uniformly to a function, then the limit function is solution to the limiting PDE. However the difficulty is in proving the convergence of the solutions. Most of the cases, we will be having the equiboundedness (or local equiboundedness) of the solutions. But this does not guarantee the convergence. Barles and Perthame introduced the concept of weak limits and showed that these weak limits satisfy the limiting PDE always. When the limiting PDE has comparison principle, they obtian that these weak limits coincide and thus obtain the existence and convergence at a single stroke. We now describe this method in detail. Here we are describing the procedure only for elliptic equation on a bounded domain Ω.

Let $F_\epsilon : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be given continuous functions such that $F_\epsilon \to F_0$ uniformly on compact sets as $\epsilon \to 0$. Consder the family of nonlinear pdes

$$F_\epsilon(x, u_\epsilon, Du_\epsilon) = 0. \tag{5.1}$$

Let u_ϵ be a viscosity solution to the equation (5.1) for each $\epsilon > 0$. Assume that u_ϵ is localy bouned uniformly in ϵ. Define the upper weak limit $\bar{u}_0$ of the sequence $\{u_\epsilon\}$ by

$$\bar{u}_0(x) = \limsup_{(y,\epsilon)\to(x,0+)} u_\epsilon(y)$$

and the lower weak limit $\underline{u}_0$ by

$$\underline{u}_0(x) = \liminf_{(y,\epsilon)\to(x,0+)} u_\epsilon(y).$$

Then we have the following lemma. We leave the proof as it is a standard real analysis exercise.

Lemma 5.1. *The upper weak limit $\bar{u}_o$ is upper semicontinuous and the lower weak limit $\underline{u}_0$ is lower semicontinuous.*

We now prove a lemma which is crucial in establishing the main result.

Lemma 5.2. *Let $B = \bar{B}(x, r)$ be a closed ball of radius r with center x. Assume that $x_0 \in B$ be a strict maximizer for $\bar{u}_0 - \phi$ on B. Then there exists a sequence $x_n \in B$ and $\epsilon_n \to 0+$ such that x_n is maximizer of $u_{\epsilon_n} - \phi$ on B, $x_{\to} x_0$ and $u_{\epsilon_n}(x_{)} \to \bar{u}_0(x_0)$. Similar statement holds for the lower weaklimit.*

Proof. We prove only for the case of upper weak limit.

Choose arbitrarily two sequences $\epsilon_n \to 0+$ and $x^n \to x_0$ such that $u_{\epsilon_n}(x^n) \to \bar{u}_0(x_0)$.

Let x_n be a maximizer of $u_{\epsilon_n} - \phi$ on B and extract subsequences, still denoted by the same, such that

$$x_n \to \bar{x}, \;\; u_{\epsilon_n}(x_n) \to \alpha.$$

Now

$$(u_{\epsilon_n} - \phi)(x_n) \geq (u_{\epsilon_n} - \phi)(x^n)$$

by definition. Hence

$$(\bar{u}_0 - \phi)(\bar{x}_0) \geq \alpha - \phi(\bar{x}) \geq (\bar{u}_0 - \phi)(x_0).$$

Since x_0 is strict maximizer, $\bar{x} = x_0$. This completes the lemma. □

We now give the main theorem.

Theorem 5.3. *The upper weak limit $\bar{u}_0$ is viscosity subsolution of (5.1) with $\epsilon = 0$ and lower weak limit is viscosiy supersolution to the same equation. Further if the equation (5.1) has comparison principle for each $\epsilon \geq 0$ and $u_\epsilon(x) \to g(x)$ on $\partial\Omega$ then $\bar{u}_o = \underline{u}_o$ is the unique viscosity solution of (5.1)) with $\epsilon = 0$ and the boundary condition $u_0 = g$ on $\partial\Omega$.*

The proof of the theorem is essentially the same as in the proof of the stability result. So we skip the details.

Disclaimer

These notes are in a very preliminary form and are intended for the benefit of the participants of the workshop. The notes depend on a lot of material, particularly on the works by Barles, Crandall, Evans, Ishii, Jensen and P.L. Lions. Also the bibliography is not complete.

References

[1] M. Bardi and I. Capuzzo Dolcetta, Optimal Control and Viscosity Solutions, Birkhauser, 1997.

[2] G. Barles, Discontiuous viscsoity solutions of first order Hamilton-Jacobi equations: a guided visit, Nonlinear Anal., 20(1993), 1123 - 1134.

[3] G. Barles, Solutions de viscosité des équations de Hamilton-Jacobi, Volume 17, Mathematiques et Applications, Springer, 1994.

[4] G. Barles and B. Perthame, Discontinuous solutions of deterministic optimal stopping time problems, RAIRO Modél. Math. Anal. Numér., 21(1987), 557 - 579.

[5] E. N. Barron and R. Jensen, The Pontryagin maximum principle from dynamic programming and viscosity solutions of first-order partial differential equations, Trans. Amer. Math. Soc., 298(1986), 635 - 641.

[6] E. N. Barron and R. Jensen, Semicontinuous viscosity solutions of Hamilton-Jacboi equations with convex Hamiltonians, CPDE, 15(1990), 1713 - 1742.

[7] M.G. Crandall, H. Ishii and P. L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. AMS, 27(1992), 1 - 67.

Industrial Engineering and Operations Research, Indian Institute of Technology Bombay, Mumbai 400 076, India

E-mail address: mallik.rao@iitb.ac.in