Semigroup Theory and Evolution Equations

Amiya Kumar Pani
Department of Mathematics
IIT Bombay, Powai
Mumbai- 400076
Email: akp@math.iitb.ac.in

1 Introduction.

In this section, we discuss scalar ODE, system of ODE, and ODEs in infine-dimensional Banach spaces.

1.1 Scalar ODE.

Simplest ODE that we come across is:

$$\begin{aligned} \frac{du}{dt} &= au, \quad t > 0, \quad a \in \mathbb{R}, \\ u(0) &= v \in \mathbb{R} \end{aligned} \tag{1.1}$$

It's solution is given by $u(t) = e^{at}\, v \quad t \geqslant 0$.
We now make some simple observations depending the real parameter a.

- If $a < 0$, every solution tends to zero as $t \to \infty$, that is, zero solution is asymptotically stable.
- In case $a = 0$, then zero solution is stable, but not asymptotically stable.
- If $a > 0$, then zero solution is unstable.

Setting $E(t) = e^{at}$, we note that $\{E(t)\}_{t\in\mathbb{R}}$ is a family of bounded linear map from $\mathbb{R}$ into itself and this family satisfies $E(0) = 1$, $E(t+s) = E(t)E(s)$ and $E(-t) = (E(t))^{-1}$. Hence, $\{E(t)\}_{t\in\mathbb{R}}$ forms a multiplicative group. Further,

$$\lim_{t\to 0} E(t) = 1 = E(0).$$

If we restrict $t \geq 0$, then $\{E(t)\}_{t\geq 0}$ forms a semi-group of bounded linear operators. Now to every DE(1.1), we attach a unique family $\{E(t)\}_{t\geqslant 0}$ of semigroup satisfying

$$E(0) = 1, \tag{1.2}$$

$$E(t+s) = E(t)E(s), \quad t, s \geq 0, \tag{1.3}$$

$$\lim_{t\to 0^+} E(t) = 1. \tag{1.4}$$

The last property is connected to the uniform continuity property of the family of semigroups.

Conversely to each family $\{E(t)\}_{t\geqslant 0}$ satisfying (1.2), we can attach an ODE (1.1), where the generator

$$\lim_{t\to 0^+} \frac{E(t)-1}{t} = a.$$

Thus, the existence, uniqueness and continuous dependence property for all time (called stability) of the solution of the ODE (1.1) is intimately connected to the family $\{E(t)\}_{t\geq 0}$ of uniformly continuous semigroup of bounded linear operators whose generator is a.

1.2 System of ODEs.

To generalize it further, consider a system of linear ODEs:

$$\begin{aligned}\frac{du}{dt} =& Au, \\ u(0) =& v \in \mathbb{R}^N,\end{aligned} \tag{1.5}$$

where for each $t \geq 0$, $u(t) \in \mathbb{R}^N$, A is $N \times N$ real matrix and $v \in \mathbb{R}^N$. This problem has a unique solution for all $t \geq 0$. Its solution can be written as $u(t) = e^{tA}\, v$. Note that

$$e^{tA} := \sum_{j=0}^{\infty} \frac{A^j t^j}{j!} \quad \text{with } A^0 = I, \tag{1.6}$$

where $I = I_{N\times N}$ identity matrix. With $E(t) = e^{tA}$, we write the solution u as $u(t) := E(t)v$. Now consider the family $\{E(t)\}_{t\geqslant 0}$. Note that if B_1 and B_2 are $N \times N$ matrices with B_1 commutes with B_2, that is $B_1B_2 = B_2B_1$, then

$$e^{t(B_1+B_2)} = e^{tB_1}\, e^{tB_2}.$$

Therefore, the semigroup property

$$E(t+s) = e^{(t+s)A} = e^{tA}\, e^{sA} = E(t)E(s)$$

is satisfied. Further, for any matrix B subordinated to a norm say $\|\cdot\|$ on $\mathcal{R}^N$,

$$\|e^{tB}\| \leq \sum_{j=0}^{\infty} \frac{\|B\|^j t^j}{j!} \leq \sum_{j=0}^{\infty} \frac{(\|B\|\; t)^j}{j!} = e^{\|B\|\; t}, \tag{1.7}$$

and hence, the family $\{E(t)\}_{t\geqslant 0}$ forms a semigroup of bounded linear operator from $\mathbb{R}^N$ to itself. Observe that this family forms an uniformly continuous semigroup $\{E(t)\}_{t\geqslant 0}$ in the sense that

$$\lim_{t\to 0^+} E(t) = I.$$

Note that its generator is

$$A = \lim_{t\to 0^+} \frac{E(t)-I}{t}.$$

Then, we can associate with a family of uniformly continuous semi-group, the solvability of the system of ODEs (1.5).

In addition, if we assume A is a real symmetric matrix, then A is diagonalizable. Let $\lambda_j, j = 1, \cdots, N$ (may be repeated) be the eigenvalues and the corresponding normalized eigenvectors be $\varphi_j,\ j = 1, \cdots, N$. Since A is symmetric, the set of eigenvectors $\{\varphi_j\}_{j=1}^N$

forms an orthonormal basis of $\mathbb{R}^N$. Then (1.5) can be written in diagonalized form. Since each $u(t)$ is a vector in $\mathcal{R}^N$, then, we can express

$$u(t) = \sum_{j=1}^{N} \alpha_j(t)\, \varphi_j,$$

where α_j $j = 1, \cdots, N$ are unknowns and can be found out from the N set of scalar ODEs:

$$\alpha_j'(t) = \lambda_j \alpha_j, \qquad j = 1...N, \tag{1.8}$$
$$\alpha_j(0) = (v, \varphi_j). \tag{1.9}$$

The solution of (1.8) can be written as $\alpha_j(t) = e^{\lambda_j t}\alpha_j(0)$. Hence,

$$u(t) = \sum_{j=1}^{N} e^{\lambda_j t}\alpha_j(0)\, \varphi_j \tag{1.10}$$
$$= \sum_{j=1}^{N} e^{\lambda_j t}(v, \varphi_j)\, \varphi_j, \tag{1.11}$$

and the semigroup $E(t)$ has a representation:

$$u(t) = E(t)v = \sum_{j=1}^{N} e^{\lambda_j t}(v, \varphi_j)\, \varphi_j. \tag{1.12}$$

If all the eigenvalues are negative, then $u(t) \to 0$ and hence, the zero solution is asymptotic stable. Further, atleast one eigenvalue is 0 and rest eigenvalues have negative real part, then zero solution is stable. In case, one eigenvalue is positive, then the zero solution. unstable.
For non-homogeneous system of linear ODE of the form:

$$\frac{du}{dt} = Au + f(t), \quad t > 0, \tag{1.13}$$
$$u(0) = v \in \mathbb{R}^N,$$

where $f(t) \in \mathcal{R}^N$. Using Duhamel's principle, we with the help of semigroup $E(t)$ obain a representation of solution as

$$u(t) := E(t)v + \int_0^t E(t-s)\, f(s)\, ds. \tag{1.14}$$

1.3 ODE in Banach Spaces.

Let X be a Banach space with norm $\|\cdot\|$. Now, consider the following evolution equation:

$$\frac{du}{dt} = Au(t), \quad t \geq 0, \tag{1.15}$$
$$u(0) = v \in X,$$

where A is a bounded linear operator on X to itself, that is, $A \in BL(X)$. Its solution u can be written as

$$u = e^{At}\, v,$$

where representation of e^{tA} is given as in (1.6). With $E(t) = e^{tA}$, as in the previous subsection we can show that the family $\{E(t)\}_{t\geqslant 0}$ forms uniformly continuous semigroup of bounded linear operators on the Banach space X.
For non-homogeneous linear ODE in Banach space X we can have exactly the same representation of solution u as in (1.14).
When X is a Hilbert space with inner-product $(\cdot,\cdot)$ and A is a selfadjoint[1], compact linear operator on X, then it has countable number of real eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$. Then consider the corresponding set of normalised eigenvectors $\{\varphi_j\}_{j=1}^{\infty}$. Indeed, $\{\varphi_j\}_{j=1}^{\infty}$ forms an orthonormal basis of X. If

$$u(t) = \sum_{j=1}^{\infty} \alpha_j(t)\varphi_j, \quad \text{where } \alpha_j(t) = (u(t), \varphi_j), \tag{1.16}$$

then using orthonormal property, we obtain the followinf infinite system of scalar ODEs:

$$\alpha_j'(t) = \lambda_j \alpha_j, \qquad \alpha_j(0) = (v, \varphi_j),$$

where $(\cdot,\cdot)$ is the inner-product on X. On solving

$$\alpha_j t = e^{\lambda_j t}\alpha_j(0).$$

Hence

$$E(t)v = u(t) = \sum_{j=1}^{\infty} e^{\lambda_j t}(v, \varphi_j)\ \varphi_j.$$

When $\|\cdot\|$ is the induced norm on X and at least one eigenvalue is zero with all are negative, then

$$\|E(t)v\| = \|u(t)\| \leq \sum_{j=1}^{\infty} \|(v, \varphi_j)\ \varphi_j\| \leq \|v\|,$$

and the solution is stable.
Below, we give an example of A as

$$Au(t) = \int_0^t K(t,s)u(s)\ ds,$$

where $K(\cdot,\cdot) \in L^2 \times L^2$ and $K(t,s) = K(s,t)$, that is, K is symmetric. With $X = L^2$, the operator $A \in BL(X)$ and A is self-adjoint. Now, we can write the solution u of (1.15) as

$$u(t) = E(t)v = e^{tA}v,$$

and we can also have a representation of u through the eigen-vectors. But when $K \in C^0 \times C^0$ and K is bounded, then with $X = C^0$ as the Banach space, we can write the solution in exponential form.
In all the above cases, $\{E(t)\}_{\ t\geqslant 0}$ is an uniformly continuous semigroup and its generator is A. Note that the solvability of (1.15) is intimately connected with the existence of a family of uniformly continuous semigruop $E(t) = e^{At}$ with its generator as $A \in BL(X)$.

[1] The bounded linear operator $A : X \longrightarrow X$ is called self-adjoint, if

$$(A\phi, \psi) = (\phi, A\psi) \quad \forall \phi, \psi \in X.$$

1.4 For more general linear operator on X.

Consider the following linear homogeneous evolution equation :

$$\begin{aligned} \frac{du}{dt} =& Au(t), \quad t > 0, \\ u(0) =& v, \end{aligned} \tag{1.17}$$

where A is a linear not necessarily bounded operaor on X with domain $D(A) \subset X$.
In this case, we can ask the following question

> Under what condition on A, it generates a semigroup of bounded linear operators on X ?

If so

> Can it have a representation like exponential type?
> Like in the previous case, does its have a relation with the solvability of the abstract evolution equation.

Some of these questions will be answered in the course of these lectures.

2 Semigroups

We begin by the definition of semigroup, and then discuss its properties. Throughout this section, assume that X is a Banach space with norm $\|\cdot\|$.

Definition 2.1. A family $\{E(t)\}_{t\geqslant 0}$ of bounded linear operators on X is said to be a Semigroup on X, if it satisfies

(i) $E(0) = I$,

(ii) $E(t+s) = E(t)E(s), \ \ t, s \geq 0.$

Definition 2.2. A linear operator A defined by

$$Av = \lim_{t\to 0^+} \frac{E(t)\ v - v}{t}$$

with its domain of definition

$$D(A) := \{v \in X : \lim_{t\to 0^+} \frac{E(t)\ v - v}{t} \text{ exists}\}$$

is called the infinitesimal generator of the family of semigroups $\{E(t)\}_{t\geqslant 0}$.

Definition 2.3. A semigroup is said to be *uniformly continuous* with respect to operator norm $\|\cdot\|$ associated with X, if

$$\lim_{t\to 0^+} \|E(t) - I\| = 0.$$

Definition 2.4. A semigroup is said to be *strongly continuous* with respect to norm $\|\cdot\|$ associated with X, if

$$\lim_{t\to 0^+} \|E(t)\ v - v\| = 0 \quad \text{for } v \in X.$$

2.1 Uniformly Continuous Semigroups.

In this subsection, we shall discuss uniformly continuous semigroups and their properties.

Theorem 2.5. *Assume that the linear operator $A \in BL(X)$. Then the family $\{E(t)\}_{t\geqslant 0}$ defined by*

$$E(t) := e^{At},$$

where

$$e^{tA} := \sum_{j=0}^{\infty} \frac{A^j t^j}{j!} \quad \text{with } A^0 = I$$

forms a uniformly continuous semigroup on X with its infinitesimal generator A.

Proof. Because of (1.7), it follows that

$$\|E(t)\| = \|e^{tA}\| \leq \sum_{j=0}^{\infty} \frac{(\|A\|\ t)^j}{j!} = e^{\|A\|\ t}, \ \ t > 0,$$

and hence, $E(t)$ is welldefined with $E(0) = I$, where I is an identity map on X. Further, it is easy to show that $E(t)$ satisfies the semigroup property in the definition 2.1 (ii). Now it remains to show the uniform continuity property. Note that for $t > 0$

$$\|E(t) - I\| \leq \sum_{j=1}^{\infty} \frac{(\|A\|\ t)^j}{j!} = e^{\|A\|\ t} - I,$$

and hence, it tends to zero as $t \to 0^+$. Further,

$$\|\frac{E(t) - I}{t} - A\| \leq \frac{1}{t} \sum_{j=2}^{\infty} \frac{(\|A\|\ t)^j}{j!} = \frac{1}{t}\Big(e^{\|A\|\ t} - I - t\|A\|\Big) \to 0, \ \ \text{as } t \to 0^+,$$

and hence, A is its infinitesimal generator with $D(A) = X$. This completes the rest of the proof. ☐

Remark 2.1. If $\{E(t)\}, t \geq 0$ is a uniformly continuous semigroup, then there are constants $\omega \geq 0$ and $M \geq 1$ such that

$$\|E(t)\| \leq Me^{\omega t}.$$

Further, for $t, s \geq 0$ there holds:

$$\lim_{s \to t} \|E(t) - E(s)\| = 0.$$

To sketch a proof, observe from the property $\lim_{t\to 0^+} \|E(t) - I\| = 0$ that for small enough $\eta > 0$ with $0 \leq s \leq \eta$ there hold: $\|E(s)\| \leq M$. Clearly, $M \geq 1$. Now setting $t = n\eta + \delta$ with $0 \leq \delta < \eta$, it follows from the semigroup property that

$$\|E(t)\| = \|E(n\eta + \delta)\| = \|E(\delta)\Big(E(\eta)\Big)^n\| \leq M^{n+1} \leq Me^{n \log M} \leq Me^{t\left(\log M/\eta\right)}.$$

Note that by choosing $\omega = \log M/\eta$, the first part of the result follows. Fo the second part, observe that for $t \geq s > 0$

$$\|E(t) - E(s)\| = \|E(s)(E(t-s) - I)\| \longrightarrow 0 \quad \text{as } t \to s,$$

and this completes the rest of the proof.

Given a bounded linear operator A on a Banach space, we can attach a uniformly continuous semigroup $E(t) = e^{tA}$ whose infinitesimal generator is A. Now let us ask: given a uniformly continuous semigroup on X, is it possible to attach a unique bounded linear operator A on X such that the given semigroup $E(t)$ is of the form e^{tA} ?
The answer is in affirmative and it is stated below in terms of a Theorem.

Theorem 2.6. *Assume the $E(t)$ is a uniformly continuous semigroup on a Banach space X. Then, there exists a unique bounded linear operator A on X such that $E(t) = e^{tA}$, for $t > 0$.*

Proof. By the property of uniformly continuous semigroup we arrive at,

$$\|E(t) - I\| \longrightarrow 0 \quad \text{as } \ t \to 0^+.$$

Now,it is observed that for small enough $\rho > 0$, there holds

$$\|\frac{1}{\rho}\int_0^\rho E(s)\, ds - I\| < 1.$$

As a consequence of von-Neumann's expansion, it follows that $\frac{1}{\rho}\int_0^\rho E(s)\, ds$ is invertible. For fixed ρ, we now claim that

$$A = \Big(E(\rho) - I\Big)\Big(\int_0^\rho E(s)\, ds\Big)^{-1}$$

is the infinitesimal generator of $E(t)$. Note that as $t \to 0^+$

$$\Big(E(t) - I\Big)\Big(\int_0^\rho E(s)\, ds\Big) = \frac{1}{t}\int_\rho^{\rho+t} E(s)\, ds - \frac{1}{t}\int_0^t E(s)\, ds \longrightarrow E(\rho) - I.$$

Thus,

$$\frac{E(t) - I}{t} \longrightarrow A, \quad \text{as } t \to 0^+.$$

and it now follows that

$$A = \Big(E(\rho) - I\Big)\Big(\int_0^\rho E(s)\, ds\Big)^{-1}.$$

For uniqueness, assume contrary, that is, the uniqueness does not hold. Thus, assume there are atleast two distinct uniformly continuous semigroups, say, $E(t)$ and $G(t)$ and both having the same infinitesimal generator A. For $t > 0$, set $\tau = t/n$. Then,

$$\begin{aligned} G(t) - E(t) &= G(n\tau) - E(n\tau) \\ &= \sum_{j=0}^{n-1} G((n-j-1)\tau)\Big(G(\tau) - E(\tau)\Big)\, E(j\tau). \end{aligned}$$

As $n \longrightarrow \infty$, $\tau \longrightarrow 0$ and further,

$$\begin{aligned} \|G(t) - E(t)\| &\leq nK(t)\|G(\tau) - E(\tau)\| \\ &= tK(t)\Big\|\frac{G(\tau) - I}{\tau} - \frac{E(\tau) - I}{\tau}\Big\| \longrightarrow 0, \ \text{as } \tau \longrightarrow 0. \end{aligned}$$

Hence, $G(t) = E(t)$ which leads to a contradiction and this completes the rest of the proof. □
Below, we establish a connection between the semigroup and the evolution equation.

Corollary 2.1. *Assume that $E(t) = e^{tA}$ is a uniformly continuous semigroup with its infinitesimal generator $A \in BL(X)$. Then,*

$$\frac{d}{dt}E(t) = AE(t) = E(t)A,$$

and $u(t) = E(t)v$ is a solution of the abstract evolution equation:

$$\frac{du}{dt} = Au, \quad t > 0 \qquad \text{with } \; u(0) = v \in X.$$

3 Strongly Continuous Semigroups.

In this section, we shall discuss strongly continuous semigroup or C^0-Semigroup, its properties and its relation to abstract evolution equation.

3.1 Properties of the Semigroup.

As in case of uniformly continuous semigroup, we have the following boundedness property.

Proposition 3.1. *Assume that $E(t)$ is a C^0-semigroup. Then there are constant $M \geq 1$ and real ω such that for $t \geq 0$*

$$\|E(t)\| \leq Me^{\omega t}.$$

Proof. As in case of uniformly continuous semi-group, we proceed to prove this exponential property provided we prove that there exist $M \geq 1$ and $\eta > 0$ such that for $0 \leq t \leq \eta$

$$\|E(t)\| \leq M.$$

Suppose this does not hold, that is, there is a sequence $\{t_m\}$ converging to zero such that $\|E(t_m)\| \geq m$ as $m \to \infty$. Since from the property of the semigroup, $E(t_m)v \longrightarrow v$ as $m \to \infty$, therefore, $\{E(t_m)v\}$ is bounded for every $v \in X$. Then, by uniform-boundedness principle (Banach-Steinhause Theorem) the sequence $\{E(t_m)\}$ is bounded which leads to a contradiction. Hence, the result. □

Theorem 3.1 (Properties)**.** *Let $E(t)$ be a C^0-semigroup and let A be its infinitesimal generator with domain $D(A)$ in X. Then the following properties hold:*

(i) For $v \in X$,

(a) the map $t \longrightarrow E(t)v$ is a continuous from $[0, \infty)$ into X.

(b) $\int_0^t E(s)v\,ds \in D(A)$ and $A\Big(\int_0^t E(s)v\,ds\Big) = E(t)v - v$.

(ii) For $v \in D(A)$

(a) $E(t)v \in D(A)$, $t \geq 0$.

(b) $AE(t)v = E(t)Av$, $t > 0$.

(c) the map $t \longrightarrow E(t)v$ is differentiable for $t > 0$.

(d) $\frac{d}{dt}(E(t)v) = AE(t)v$, $t > 0$.

Proof. For (i) (a), note that for any $\tau \geq 0$

$$\begin{aligned} \|E(t+\tau)v - E(t)v\| &\leq \|E(t)\| \, \|E(\tau)v - v\| \\ &\leq Me^{\omega t} \, \|E(\tau)v - v\|. \end{aligned}$$

Then as $\tau \longrightarrow 0$ the result follows for the right hand limit. Similarly, we can argue for left hand limit and then result follows.
For (i) (b), by the continuity of $E(t)$ we observe that for $h \longrightarrow 0$

$$\frac{(E(h)-I)}{h} \int_0^t E(s)v \, ds = \frac{1}{h}\Big(\int_t^{t+h} E(s)v \, ds - \int_0^h E(s)v \, ds\Big) \longrightarrow E(t)v - v,$$

and the result follows.
Now for (ii) $(a)-(b)$ since $v \in D(A)$, then using semigroup property

$$\begin{aligned} \lim_{h \longrightarrow 0^+} \frac{E(h)\ E(t)v - E(t)v}{h} &= \lim_{h \longrightarrow 0^+} \frac{E(t)\ E(h)v - E(t)v}{h} \\ &= E(t) \lim_{h \longrightarrow 0^+} \frac{E(h)v - v}{h} \\ &= E(t)\ Av. \end{aligned}$$

Hence, $E(t)v \in D(A)$ and $A\ E(t)v = E(t)\ Av$.
Now for (ii) $(c)-(d)$, observe that for $v \in D(A)$,$h > 0$ and for $t > 0$

$$\begin{aligned} \lim_{h \longrightarrow 0^+} \quad & \Big(\frac{E(t)v - E(t-h)v}{h} - E(t)\ Av\Big) \\ &= \lim_{h \longrightarrow 0^+} \Big(E(t-h)\Big(\frac{E(h)v - v}{h}\Big) - E(t)\ Av)\Big) \\ &= \lim_{h \longrightarrow 0^+} \Big(E(t-h)\Big(\frac{E(h)v - v}{h} - Av\Big) - (E(t-h) - E(t))\ Av\Big) \longrightarrow 0, \end{aligned}$$

since $\frac{E(h)v-v}{h} \longrightarrow Av$ and the semigroup is bounded. Thus,

$$\lim_{h \longrightarrow 0^+} \frac{E(t)v - E(t-h)v}{h} = E(t)\ Av.$$

Similarly, it is easy to check that

$$\lim_{h \longrightarrow 0^+} \frac{E(t+h)v - E(t)v}{h} = E(t)\ Av.$$

Therefore, $\frac{d}{dt}E(t)v$ exists and is equal to $E(t)\ Av$ for $v \in D(A)$. This completes the rest of the proof. ☐

Problem 3.1. *Show that for $t > s \geq 0$ and for $v \in D(A)$*

$$E(t)v - E(s)v = \int_s^t E(\tau)\ Av \, d\tau.$$

Using this result prove that the infinitesimal generator A defines the semigroup $E(t)$ uniquely.

Now we state an important Theorem regarding solvability of the abstract evolution equation.

Theorem 3.2. *Let A is the infinitesimal generator of C^0-semigroup $\{E(t),\ t \geq 0\}$ on X with domain $D(A) \subset X$. Then for $v \in D(A)$, $u(t) = E(t)v$ defines a unique solution of the abstract evolution problem:*

$$\frac{du}{dt} = Au, \quad t \geq 0, \tag{3.18}$$

$$u(0) = v \tag{3.19}$$

satisfying $u \in C^0([0,\infty); D(A)) \cap C^1([0,\infty); X)$.

Proof. Let $v \in D(A)$ and define $u(t) := E(t)v$. By the Theorem 3.1 (ii) (b), it follows that

$$AE(t)v = E(t)Av$$

and by (ii) (c), the mapping

$$t \longrightarrow E(t)Av$$

is continuously differentiable from $[0,\infty)$ into $D(A)$. Further,

$$\frac{d}{dt}E(t)v = AE(t)v = E(t)Av,$$

and hence, $u(t)$ satisfies the abstract evolution equation (3.18) with initial condition $u(0) = v$.

For uniqueness, assume contrary, that is, the solution is not unique. Let u and w be two distinct solutions of the problem (3.18). Define $y(t)$ as

$$y(s) = E(t-s)w(s), \quad 0 \leq s \leq t.$$

Then,

$$\frac{dy}{ds} = -AE(t-s)w(s) + E(t-s)Aw(s) = 0.$$

and therefore, $y(s) = y(0), \quad s \in [0,t]$. In particular, $y(t) = w(t)$ and $y(0) = u(t)$. Hence, $w(t) = u(t)$ for all $t \geq 0$ which leads to a contradiction. Therefore, the solution is unique and this concludes the proof. □

Remark 3.1. Note that if $v \notin D(A)$, then $E(t)v$ is not differentiable with respect to time. However, $u(t) = E(t)v$ with $v \in X$ can be thought of a generalized solution of (3.18).

Below, we discuss the properties of the generator.

Theorem 3.3 (Properties of the generator)**.** *Let A be the infinitesimal generator of a C^0-semigroup $\{E(t)\}$. Then, $D(A)$ is dense in X and A is closed.*

Proof. For any $v \in X$, from theorem 3.1 (i) (b), $\int_0^t E(s)v\,ds \in D(A)$. Setting $v_t = \frac{1}{t}\int_0^t E(s)v\,ds$, then each $v_t \in D(A)$, $t > 0$. Since $E(s)v \longrightarrow v$ as $s \to 0$, therefore, $v_t \longrightarrow v$. This implies that $D(A)$ is dense in X.

To complete the rest of the proof, we need to prove that the operator A is closed. Consider any sequence $\{v_n\}$ in $D(A)$ with $v_n \to v$ in X and $Av_n \to w$ in X, we now claim that $v \in D(A)$ and $w = Av$. Note that by the above argument:

$$\frac{E(t)v_n - v_n}{t} = \frac{1}{t}\int_0^t E(s)Av_n\,ds,$$

and hence taking limit of both sides as $n \to \infty$, we arrive at

$$\frac{E(t)v - v}{t} = \frac{1}{t}\int_0^t E(s)w\, ds.$$

Now taking $t \longrightarrow 0$ in both sides, it follows that $v \in D(A)$ and $Av = w$. This now completes the proof. □

3.2 Hille-Yosida Theorem

Note that for solvability of the abstract evolution equation (3.18)-(3.19), it is more pertinent to ask the following question:

Under what conditions on the operator A, it generates a C^0-semigroup ?

The answer to the above question is given by the Hille-Yosida Theorem.
Below, we give some definitions for our future use.

Definition 3.4. The semigroup $E(t)$ is called a contraction semigroup if $\|E(t)\| \leq 1$ for all $t \geq 0$.

Definition 3.5. The resolvent set $\rho(A)$ of the operator A is defined as

$$\rho(A) = \{z \in \mathbb{C} : R(z; A) = (zI - A)^{-1} \text{exits and bounded}\},$$

and $R(z; A)$ is called resolvent operator associated with A.

Below, we state the main theorem of this subsection.

Theorem 3.6 (Hille-Yosida Theorem). *A linear operator A on X with $D(A) \subset X$ is the infinitesimal generator of a C^0-semigroup of contraction $E(t)$ with $\|E(t)\| \leq 1$ if and only if*

(i) A is closed and $D(A)$ is dense in X.

(ii) $\rho(A) \supset (0, \infty)$ and $\|R(\lambda; A)\| \leq \frac{1}{\lambda}$ for $\lambda > 0$.

For the proof of the above theorem, we require some properties of Resolvent operator, which are given below.

Lemma 3.1 (Properties of Resolvent Operator). *Let A be the infinitesimal generator of a strongly continuous semigroup $E(t)$ of contraction on X. Then the following properties hold:*

(i) (Resolvent Identity). For real λ and μ in $\rho(A)$,

$$R(\lambda; A) - R(\mu; A) = (\mu - \lambda)\, R(\lambda; A)\, R(\mu; A),$$

and

$$R(\lambda; A)\, R(\mu; A) = R(\mu; A)\, R(\lambda; A).$$

(ii) For $\lambda \in \rho(A)$ and $v \in X$, then $\lambda > 0$,

$$R(\lambda; A)\ v := \int_0^\infty e^{-\lambda s} E(s)\ v\ ds,$$

and

$$\|R(\lambda; A)\| \leq \frac{1}{\lambda}.$$

Proof. From the definition, it is easy to show (i). For (ii), define for $v \in X$,

$$R(\lambda)v = \int_0^\infty e^{-\lambda s} E(s)\ v\ ds.$$

Since it is a contraction semigroup and $\lambda > 0$, the integral is well-defined. Moreover, the mapping $v \longrightarrow R(\lambda)v$ is a linear map on X and

$$\|R(\lambda)v\| \leq \|v\| \int_0^\infty e^{-\lambda s}\ ds \leq \frac{1}{\lambda}\ \|v\|.$$

This is a bounded linear operator with

$$\|R(\lambda)\| \leq \frac{1}{\lambda}, \quad \lambda > 0.$$

Thus, to complete the proof, it remains to show that

$$R(\lambda) = R(\lambda; A), \quad \lambda > 0.$$

Now for $h > 0$ and $v \in X$, note that by definition

$$\begin{aligned}
\Big(\frac{E(h) - I}{h}\Big)\ R(\lambda)v &= \frac{1}{h}\int_0^\infty e^{-\lambda s}(E(s+h)v - E(s)v)\ ds \\
&= \frac{1}{h}\int_0^\infty e^{-\lambda(s-h)} E(s)v\ ds - \frac{1}{h}\int_0^\infty e^{-\lambda s} E(s)v\ ds \\
&= \Big(\frac{e^{\lambda h} - 1}{h}\Big)\int_0^\infty e^{-\lambda s} E(s)v\ ds - \frac{e^{\lambda h}}{h}\int_0^\infty e^{-\lambda s} E(s)v\ ds \\
&\longrightarrow \lambda R(\lambda)v - v,
\end{aligned}$$

as $h \to 0$. Hence, $R(\lambda)v \in D(A)$ and

$$AR(\lambda)v = \lambda R(\lambda)v - v.$$

Rewrite it as

$$(\lambda I - A)R(\lambda)v = v, \quad \text{for all } v \in X.$$

Now, for all $v \in D(A)$, observe that

$$\begin{aligned}
R(\lambda)Av &= \int_0^\infty e^{-\lambda s} E(s)Av\ ds = \int_0^\infty e^{-\lambda s}\frac{d}{ds}(E(s)v)\ ds \\
&= \lambda\int_0^\infty e^{-\lambda s} E(a)v\ ds - v \\
&= \lambda R(\lambda)v - v,
\end{aligned}$$

and hence,

$$R(\lambda)\ (\lambda I - A)v = v, \quad \text{for all } v \in D(A).$$

Thus,

$$R(\lambda) = (\lambda I - A)^{-1},$$

and this completes the rest of the proof. □

Lemma 3.2. *If A is defined as in Lemma 3.1, then there holds for all $v \in X$*

$$\lim_{\lambda \longrightarrow \infty} \lambda R(\lambda; A)v = v.$$

Proof. Let us first prove the result for $v \in D(A)$. Using the definition and properties of the Resolvent operator, it follows that

$$\|\lambda R(\lambda; A)v - v\| = \|AR(\lambda; A)v\| = \|R(\lambda; A)Av\| \leq \frac{1}{\lambda}\|Av\|,$$

and as $\lambda \to \infty$ this leads to zero.
Since $D(A)$ is dense in X, for any $v \in X$, there exists a sequence $\{v_n\}$ in $D(A)$ such that

$$\begin{aligned} \|\lambda R(\lambda; A)v - v\| &\leq \|\lambda R(\lambda; A)(v - v_n)\| + \|\lambda R(\lambda; A)v_n - v_n\| + \|v_n - v\| \\ &\leq 2\|v_n - v\| + \|\lambda v_n - v_n\|, \end{aligned}$$

and this tend to zero as $n \to \infty$ and then $\lambda \to \infty$. Now, the result follows and this concludes the proof. □

In the sequel, we define a sequence of bounded linear operators on X called Yosida approximations, which approximate the operator A.

Definition 3.7. For $\lambda > 0$, the Yosida approximation A_λ of A is defined by

$$A_\lambda := \lambda AR(\lambda; A) = \lambda^2 R(\lambda; A) - \lambda I.$$

Lemma 3.3. *For $v \in D(A)$, there holds*

$$\lim_{\lambda \to \infty} A_\lambda v = Av.$$

Proof. Using the definition of Yosida approximation and Lemma 3.2, we note that for $v \in D(A)$

$$\lim_{\lambda \to \infty} A_\lambda v = \lim_{\lambda \to \infty} \lambda AR(\lambda; A)v = \lim_{\lambda \to \infty} \lambda R(\lambda; A)Av = Av.$$

This concludes the proof. □

Lemma 3.4. *The Yosida approximation A_λ is the infinitesimal generator of a uniformly continuous semigroup $\{E_\lambda(t) := e^{tA_\lambda}\}$ of contraction and for $\lambda, \mu > 0$*

$$\|E_\lambda(t)v - E_\mu(t)v\| = \|e^{tA_\lambda}v - e^{tA_\mu}v\| \leq t\|A_\lambda v - A_\mu v\|, \quad t \geq 0.$$

Proof. From the definition of Yosida approximation

$$\|e^{tA_\lambda}\| = e^{-\lambda t}\|e^{(t\lambda^2 R(\lambda; A))}\| \leq e^{-\lambda t}\ e^{(t\lambda^2 \|R(\lambda; A)\|)} \leq e^{-\lambda t}\ e^{\lambda t} = 1,$$

and it is a contraction.

For $\lambda, \mu > 0$, the resolvent operators $R(\lambda; A)$ and $R(\mu; A)$ commute, so also Yosida approximations A_λ,A_μ and the corresponding semigroups $E_\lambda(t)$, $E_\mu(t)$. Note that

$$A_\mu \, E_\lambda(t) = E_\lambda(t) A_\mu$$

and for $v \in X$, using semigroup property

$$\frac{d}{dt} E_\lambda(t)v = A_\lambda E_\lambda(t)v = E_\lambda(t) A_\lambda v.$$

Then for $v \in X$,

$$\begin{aligned} E_\lambda(t)v - E_\mu(t)v &= \int_0^t \frac{d}{ds}\Big(E_\mu(t-s)E_\lambda(s)v\Big)\, ds \\ &= \int_0^t E_\mu(t-s)E_\lambda(s)\Big(A_\mu v - A_\lambda v\Big)\, ds, \end{aligned}$$

and hence,

$$\|E_\lambda(t)v - E_\mu(t)v\| \leq \int_0^t \|E_\mu(t-s)\| \, \|E_\lambda(s)\| \, \|A_\mu v - A_\lambda v\| \, ds \leq t \|A_\mu v - A_\lambda v\|,$$

and this completes rest of the proof. □

Proof of Theorem 3.5.
Necessary Condition. Assume that A is the infinitesimal generator of a C^0- semigroup $E(t)$ of contraction. We now claim that (i)-(ii) hold. As a consequence of Theorem 3.2, the condition (i) is easy to show. Now for (ii), we apply the proof of Lemma 3.1 (ii) to conclude the result.
Sufficient Condition. Assuming (i)-(ii) to hold for the linear operator A, we show that it generates a C^0- semigroup $E(t)$ of contraction. From A, we now construct Yosida approximations for $\lambda > 0$ as A_λ. Since each A_λ is bounded linear operator on X, it generates uniformly continuous semigroup $E_\lambda(t)_{t \geq t \geq 0} = e^{tA_\lambda}$.
From Lemma 3.4, we note that as $\lambda, \mu \to \infty$,

$$\|E_\lambda(t)v - E_\mu(t)v\| \leq t\|A_\lambda v - A_\mu v\|, \;\; t \geq 0.$$

tends to zero. Hence, we define, $E(t)v$ as

$$E(t)v = \lim_{\lambda \to \infty} E_\lambda(t)v, \;\; t \geq 0, \quad v \in D(A) \tag{3.20}$$

Observe that $E(t)v$ exists for all $v \in D(A)$ and for $t \geq 0$. Since $\|E_\lambda(t)\| \leq 1$, using denseness property of $D(A)$ in X, it follows that (3.20) holds for all $v \in D(A)$, uniformly for t on compact subsets of $[0, \infty)$. Now it is easy to verify that $E(t)$ is a C^0-semigroup of contraction.
To complete the rest of the proof, it remains to show that given linear operator A is the infinitesimal generator of $E(t)$. Write the generator of $E(t)$ as the operator B, that is, to show that $B = A$.
Observe that

$$E_\lambda(t)v - v = \int_0^t E_\lambda(s) A_\lambda v \, ds, \tag{3.21}$$

and for $v \in D(A)$ with the help of Lemmas 3.3-3.4

$$\|E_\lambda(s)A_\lambda v - E(s)Av\| \leq \|E_\lambda(s)\| \, \|A_\lambda v - Av\| + \|(E_\lambda(s) - E(s))Av\| \longrightarrow 0$$

as $\lambda \to \infty$. Hence, passing limit in (3.21) as $\lambda \to \infty$, we arrive at for $v \in D(A)$

$$E(t)v - v = \int_0^t E(s)Av \, ds,$$

and thus, $D(A) \subseteq D(B)$. Note that for $v \in D(A)$

$$Bv = \lim_{t \to 0^+} \frac{E(t)v - v}{t} = Av.$$

Since $\rho(A) \supset (0, \infty)$, $1 \in \rho(A)$ and hence,

$$(I - B)(D(A)) = (I - A)(D(A)) = X.$$

Therefore, by the necessity part of the theorem, it follows that $1 \in \rho(B)$, and

$$D(B) = (I - B)^{-1}X = D(B)$$

proving that $A = B$. This concludes the proof. □

Remark 3.2. We can state a general Hille-Yosida Theorem with out proof. For a proof, one can check Pazy [3].

Theorem 3.8 (Hille-Yosida Theorem)**.** *A linear operator A on X with $D(A) \subset X$ is the infinitesimal generator of a C^0-semigroup $E(t)$ with $\|E(t)\| \leq Me^{\omega t}$, for some $M \geq 1$ and for some real ω if and only if*

- *(i) A is closed and $D(A)$ is dense in X.*
- *(ii) $\rho(A) \supset (\omega, \infty)$ and $\|R(\lambda; A)\| \leq \frac{M}{(\lambda - \omega)^n}$ for $\lambda > \omega, \; n \geq 1$.*

When $A \in BL(X)$, then we write an exponential formula for the semigroup. Then, one is curious to know ' If A is unbounded linear operator, whether it is possible to write an exponential formula for the semigroup whose infinitesimal generator is A. Obviously, the exponential formula given through the infinite sum will land in difficulties and on the other hand

$$E(t)v := \lim_{\lambda \to \infty} e^{tA_\lambda} v, \;\; t \geq 0.$$

Therefore, one way to attach a meaning it through the expression

$$E(t)v := \lim_{n \to \infty} \Big(I - \frac{t}{n}A\Big)^{-n} v = \lim_{n \to \infty} \Big(\frac{n}{t}R(\frac{n}{t}; A)\Big)^n v.$$

For a proof, see pp. 184-185 of Kesavan [2]

3.3 Lumer-Phillips Theorem

When X is a Hilbert space with innerproduct $(\cdot,\cdot)$, we have easily verifiable conditions on the linear operator A which generates C^0-semigroup of contraction.

Definition 3.9. An operator $A : D(A) \subset X \longrightarrow X$ is said to be dissipative if

$$\Re e(Au, u) \leq 0, \quad \text{for all } u \in D(A).$$

Below, we state without proof the Lumer-Phillips Theorem. For a proof, see, Pazy[3].

Theorem 3.10 (Lumer Phillips Theorem). *Let $A : D(A) \subset X \longrightarrow X$ be a densely defined operator.*

(i) If A is dissipative and range of $(\lambda_0 I - A)$ is the whole of X for at least one $\lambda_0 > 0$, then A generates a C^0-semigroup $E(t)$ of contractions.

(ii) If A is the infinitesimal generator of C^0-semigroup $E(t)$ of contractions, then range of $(\lambda I - A)$ is the whole of X for all $\lambda > 0$ and A is dissipative.

3.4 Analytic Semigroups.

Often, we shall be using the definition of an analytic semigroup or the operator A being sectorial on X. Therefore, in this subsection, a part from the definition, we discuss some properties of analytic semigroup.
An operator $A : D(A) \subset X \to X$ is called sectorial operator on X, if A is densely defined closed operator on X, whose resolvent $R(z; A)$ is analytic in a sector :

$$\Sigma_\delta := \{z \neq 0 : |\arg z| < \delta \;\text{ with }\; \delta \in (\frac{\pi}{2}, \pi)\},$$

and bounded by

$$\|R(z; A)\| \leq \frac{M}{|\lambda|} \;\forall z \in \Sigma_\delta, \text{for some } M > 0,\; \delta \in (\frac{\pi}{2}, \pi). \tag{3.22}$$

The semigroup $E(t)$ generated by the generator A is called an analytic semigroup.
Note that (see, Pazy [3])

$$E(t) = \frac{1}{2\pi i} \int_\Gamma e^{tz}\, R(z; A)\, dz \tag{3.23}$$

where the contour Γ may be taken as a suitable path in Σ_d from $\infty e^{-i\psi}$ to $\infty e^{i\psi}$ for $\psi \in (\frac{\pi}{2}, \delta)$, that is,

$$\Gamma = \{z : \arg z| = \psi \in (\frac{\pi}{2}, \delta)\}.$$

Observe that on differentiating (3.23), we obtain

$$E'(t) = \frac{1}{2\pi i} \int_\Gamma z\, e^{tz}\, R(z; A)\, dz, \tag{3.24}$$

and hence,

$$\begin{aligned} \|E'(t)\| &\leq \frac{1}{2\pi i} \int_\Gamma |z| e^{-t\Re e z} ||R(z; A)|dz| \qquad (3.25) \\ &\leq K \int_0^\infty e^{-ct}\, ds = \frac{K}{t}. \end{aligned}$$

Note that

$$\|E(t)\| + t\|E'(t)\| \leq K. \tag{3.26}$$

3.5 Nonhomogeneous Evolution Equations.

In this subsection, we shall discuss the non-homogeneous abstract evolution equations. Given the infinitesimal generator A of a C^0-semigroup $\{E(t)\}_{t\geq 0}$ on a Banach space X with domain $D(A) \subset X$, a function $u_0 \in X$ and a mapping $f : [0,T] \longrightarrow X$, consider the following non-homogeneous abstract evolution problem:

$$\frac{du}{dt} = Au + f, \quad t \geq 0, \tag{3.27}$$

$$u(0) = u_0. \tag{3.28}$$

4 Applications.

In this section, we shall discuss some applications to evolution equations.

Example 4.1. Consider the 1st order linear PDE with Cauchy data:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x}, \quad t > 0, x \in \mathbb{R}, \\ u(0) &= v, \quad x \in \mathbb{R}. \end{aligned} \tag{4.29}$$

With $X = L^2(\mathbb{R})$, choose $A\varphi = \frac{d\varphi}{dx}$ with its domain $D(A) = H^1(\mathbb{R})$. Then, $(A\varphi, \varphi) = (\varphi', \varphi) = -(\varphi, \varphi') = -(\varphi', \varphi) = -(A\varphi, \varphi)$. Thus $\Re e(A\varphi, \varphi) = 0$ and the operator A is dissipative. Now we claim that the Range of $(\lambda I - A)$ is X, that is, for given $f \in L^2(\mathbb{R})$, there is a unique solution of

$$\lambda\varphi - \varphi' = f, \qquad \lambda > 0.$$

Note that using integrating factor, it follows that

$$-(e^{-\lambda x}\varphi)' = (e^{-\lambda x} f(x)$$

and on integrating from $-\infty$ to x, we arrive at

$$\varphi(x) = \int_{-\infty}^{x} e^{\lambda(x-s)} f(s)\, ds.$$

Thus, for $\lambda > 0$, the Range of $(\lambda I - A)$ is the whole of $X = L^2(\mathbb{R})$.
Therefore, A generates a C^0-semigroup of contraction $E(t)$ and $u(t) = E(t)v$. This also establish the solvability of (4.29). In this case, it is easy to write down $E(t)v$ explicitly as $E(t)v(x) = v(x-t)$.

Observe that it is not easy to verify the conditions specially the condition (ii) of the Hille-Yosida Theorem.

Example 4.2. Consider

$$\begin{aligned} u_t &= \Delta u, \quad t > 0, x \in \mathbb{R}^d \\ u(x,0) &= v(x), \quad x \in \mathbb{R}^d, \end{aligned} \tag{4.30}$$

To discuss its solvability, choose $X = L^2(\mathbb{R})$ and $A = \Delta$, with its domain $D(A) = H^2(\mathbb{R}^d)$. With $t \longrightarrow u(t) \in X$, we can write (4.30) in abstract form as in (1.17). Now for $\varphi \in D(A)$,

$$(A\varphi, \varphi) = (\Delta\varphi, \varphi) = -||\nabla\varphi||^2 \leq 0,$$

so that the operator A is dissipative. We claim that for $\lambda > 0$ the Range of $(\lambda I - A)$ is the whole of $L^2(\mathbb{R}^d)$. For $f \in L^2(\mathbb{R}^d)$, consider the problem :

$$\lambda u - \Delta u = f, \quad x \in (\mathbb{R}^d), \quad \lambda > 0. \tag{4.31}$$

Note that

$$(\nabla u, \nabla \chi) + \ \lambda(u, \chi) = (f, \chi) \quad \forall \chi \in H^1(\mathbb{R}^d).$$

By Lax- Milgram Theorem, the above problem has a unique solution $u \in H^1(\mathbb{R}^d)$ for a given $f \in L^2(\mathbb{R}^d)$. More over using Fourier tranform and Plancheral's identity, we can show that (4.31) has a unique solution $u \in H^2(\mathbb{R}^d)$ for $f \in L^2(\mathbb{R}^d)$. Hence,

$$\text{Range } (\lambda I - A) = L^2(\mathbb{R}^d),$$

and A generates C^0-semigroup of contraction. Moreover, it completes the solvability of the abstract problem. Here, using Fourier transform, one can write the semigroup as

$$E(t)v(x) := \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-|x-y|^2/4t} v(y)\, dy.$$

If v has compact support, then $u(x,t) = E(t)v(x)$ is non-zero for all $x \in \mathbb{R}^d$, when $t > 0$. This is known as infinite speed of propagation of solution. We now observe that when v has a compact support, then $u(x,t) \longrightarrow 0$ exponentially as $|x| \longrightarrow \infty$. Therefore, the effect at large distances although nonzero is negligible.

Example 4.3. consider the Schrodinger equation:

$$u_t = i\Delta u, \quad t > 0, \quad x \in \mathbb{R}^d \tag{4.32}$$

with initial condition $u(0) = v$. With $X = L^2(\mathbb{R}^d)$, set $A\varphi = i\Delta\varphi$ and $D(A) = H^2(\mathbb{R}^d)$. In order to apply the Lumer-Phillip's theorem, we need to check the A is dissipative, that is, for $v \in D(A)$,

$$(Av, v) = -i\|\nabla v\|^2,$$

and

$$\Re e(Av, v) = 0.$$

Therefore, it remains to show Range $(\lambda I - A) = L^2(\mathbb{R}^d),\ u \in D(A),\ \ \lambda > 0$. Now for $f \in L^2(\mathbb{R}^d)$, we need to find a unique solution $v \in H^2(\mathbb{R}^d)$ of the problem:

$$\lambda v - i\Delta v = f.$$

As has been done earlier, we apply the Lax-Milgram Lemma to infer the existence of a unique solution $v \in H^1(\mathbb{R}^d)$ and use Fourier transformation technique to infer that $v \in H^2(\mathbb{R}^d)$. This completes the rest of the proof.

Example 4.4. Let X be a Hilbert space with inner-product $(\cdot, \cdot)$ with norm $\|\cdot\|$. Consider the abstract evolution equation :

$$\begin{aligned} \frac{du}{dt} =& Au, \quad t > 0, \\ u(0) =& v, \end{aligned} \tag{4.33}$$

where $A : D(A) \subset X \longrightarrow X$, $v \in X$ and the map $t \longrightarrow u(t) \in X$. Here, we assume that $-A$ is self-adjoint, positive definite linear operator with compact inverse. Therefore, there is an orthonormal basis of eigen-functions $\{\varphi_j\}_{j=1}^{\infty}$ and corresponding eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ with

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_j \leq \ldots \quad \text{with } \lambda_j \longrightarrow \infty.$$

Hence, for any $w \in X$, there holds generalized Fourier expansion

$$w = \sum_{j=1}^{\infty} (w, \varphi_j)\varphi_j \quad \text{and} \quad -Aw = \sum_{j=1}^{\infty} \lambda_j (w, \varphi_j)\varphi_j.$$

Setting

$$u(t) := \sum_{j=1}^{\infty} u_j(t)\varphi_j,$$

where the generalized Fourier coefficient $u_j(t)$ is given by $u_j(t) = (u(t), \varphi_j)$. Forming inner-product between (4.33) and φ_j yields infinite number of scalar ODEs:

$$\frac{du_j}{dt} + \lambda_j u_j = 0, \quad t > 0 \quad \text{with} \;\; u_j(0) = v_j, \tag{4.34}$$

where $v_j = (v, \varphi_j)$. On solving, we obtain

$$u_j(t) = e^{-\lambda_j t}\, v_j,$$

and hence,

$$u(t) = E(t)v := \sum_{j=1}^{\infty} e^{-\lambda_j t}(v, \varphi_j)\varphi_j.$$

This is a C^0 semigroup as per the Lumer-Phillips Theorem and we note that by Parseval's identity

$$\begin{aligned} \|E(t)v\|^2 &= \sum_{j=1}^{\infty} e^{-2\lambda_j t}(v, \varphi_j)^2 && (4.35) \\ &\leq e^{-\lambda_1 t} \sum_{j=1}^{\infty} (v, \varphi_j)^2 = e^{-\lambda_1 t}\|v\|^2 \leq \|v\|^2. && (4.36) \end{aligned}$$

Hence, it is a C^0-semigroup of contraction. Further, observe that

$$E'(t)v = AE(t)v = -\sum_{j=1}^{\infty} \lambda_j e^{-t\lambda_j}(v, \varphi_j)\Phi_j,$$

and hence,

$$\begin{aligned} \|E'(t)v\|^2 &= \sum_{j=1}^{\infty} \lambda_j^2 e^{-2t\lambda_j}(c, \Phi_j)^2 \\ &\leq \sup_j (\lambda_j^2 t^2 e^{-2t\lambda_j}) \frac{1}{t^2} \sum_{j=1}^{\infty} (v, \varphi_j)^2. \end{aligned}$$

With $C^2 = \sup_j(\lambda_j^2 t^2 e^{-2t\lambda_j})$, we now arrive at

$$\|E'(t)v\| \leq \frac{C}{t}\|v\|, \tag{4.37}$$

and this is called smoothing property as

$$\|E'(t)v\| = \|AE(t)v\| = \|Au(t)\| \leq \frac{C}{t}\|v\|.$$

In this case, the resolvent operator $R(z;A)v$ has the representation as

$$R(z;A)v = (zI - A)^{-1}v = \sum_{j=1}^{\infty}\Big(\frac{1}{z+\lambda_j}\Big)(v,\varphi_j)\varphi_j.$$

Now if $z \in \Sigma_\delta,\ \ \delta \in (\pi/2, \pi)$, then we obtain

$$\|R(z;A)\| = \sup_j \frac{1}{|z+\lambda_j|} \leq \frac{C}{|z|}, \tag{4.38}$$

as $|z+\lambda_j| \geq |z|, \quad$ if $\Re ez \geq 0$, and if $\Re ez < 0$, the it is greater than $|\Im z| \geq (\sin\delta)^{-1}|z|$. Therefore, A is sectorial and $\{E(t)\}$ is an analytic semi-group.
To provide an concrete example, consider the following linear parabolic problem: Find $u = u(x,t)$ such that

$$\begin{aligned} \frac{\partial u}{\partial t} &= Au, \quad x \in \Omega, \quad t > 0, && (4.39)\\ u(x,t) &= 0, \quad x \in \partial\Omega, \quad t > 0, && (4.40)\\ u(x,0) &= v, \quad x \in \Omega, && (4.41) \end{aligned}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with smooth boundary $\partial\Omega$ and the operator A is defined as

$$-A\phi := -\sum_{j,k=1}^{d} \frac{\partial}{\partial x_j}\Big(a_{jk}\frac{\partial \phi}{\partial x_k}\Big) + \sum_{j=1}^{d} b_j \frac{\partial \phi}{\partial x_j} + a_0\phi. \tag{4.42}$$

Assume that

- the coefficients a_{jk}, b_j, a_0 are smooth and bounded with $a_{jk} = a_{kj}$, $\nabla \cdot b = 0$ and $a_0 > 0$, where $b = (b_1, \cdots, b_d)$.
- the operator $-A$ is uniformly elliptic, that is, there exists $\alpha_0 > 0$ such that

$$\sum_{k=1}^{d}\sum_{j=1}^{d} a_{jk}\xi_j\ \xi_k \geq \alpha_0|\xi|^2, \quad 0 \neq \xi \in \mathbb{R}^d.$$

With $X = L^2(\Omega)$ with innerproduct $(\cdot,\cdot)$ and $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, we note that

$$(-A\phi, \phi) \geq \alpha_0\|\phi\|^2_{H_0^1(\Omega)} \quad \text{for all } \phi \in H_0^1(\Omega). \tag{4.43}$$

Observe that $D(A)$ is dense in X and from (4.43), it follows that A is dissipative. Moreover, we need to verify that for a fixed $\lambda_0 > 0$, the Range of $(\lambda_0 - A) = X$, that is, for fixed $\lambda_0 > 0$ and $f \in X = L^2(\Omega)$, the following elliptic problem:

$$\begin{aligned} -Aw + \lambda_0 w &= f \quad \text{in } \Omega,\\ w &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

has a unique solution $w \in D(A) := H^2(\Omega) \cap H_0^1(\Omega)$. Using the Lax-Milgram Lemma, it is easy to check since $-A$ satisfies coercivity (4.43) condition that the unique weak solution $w \in H_0^1(\Omega)$. Then by elliptic regularity, it follows that $w \in H^2(\Omega) \cap H_0^1(\Omega)$ and hence, an application of Lummer-Phillips Theorem yields the existence of C^0-semigroup $E(t)$ of contraction,whose generator is A and the resolvent operator $R(\lambda; A)$ satisfies

$$\|R(\lambda; A)\| \leq \frac{1}{\lambda}, \quad \lambda > 0.$$

It can be shown that $E(t)$ generates an analytic semigroup on $X = L^2(\Omega)$.
If $b_j = 0,\ j = 1, \cdots, d$, the corresponding operator $-A$ is self-adjoint, that is,

$$(-A\phi, \psi) = (\phi, -A\psi) \ \ \forall \phi, \psi \in D(A),$$

and positive definite. Moreover, $-A$ has a compact inverse, which can be checked from the elliptic theory, see, [2] and [1]. So we can have a countable eigen-values $\{\lambda_j\}_{j=1}^{\infty}$ with $\lambda_{j+1} \geq \lambda_j \geq \cdots > \lambda_1 > 0$ and the corresponding eigenvectors $\{\varphi_j\}_{j=1}^{\infty}$ forms an orthonormal basis of X. Therefore, using generalized Fourier expansion, it follows that

$$u(x,t) = E(t)v := \sum_{j=1}^{\infty} e^{-t\lambda_j}(v, \varphi_j)\varphi_j.$$

Problem 4.1. *Show that the solution decays exponentially.*

Example 4.5. Second Order Hyperbolic Equations. Consider $u(x,t)$ satisfying

$$\begin{aligned} u_{tt} &= Lu \quad \text{in } \Omega \times (0,\infty), && (4.44)\\ u(x,t) &= 0 \quad \text{on } \partial\Omega \times (0,\infty), && (4.45)\\ u(x,0) &= g, \quad u_t(x,0) = h \quad \text{in } \Omega, && (4.46) \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial\Omega$ and the operator A is given by

$$-L\phi := -\sum_{j,k=1}^{d} \frac{\partial}{\partial x_j}\Big(a_{jk}\frac{\partial \phi}{\partial x_k}\Big) + a_0\phi. \tag{4.47}$$

Assume that

- the coefficients a_{jk}, b_j, a_0 are smooth and bounded with $a_{jk} = a_{kj}$ and $a_0 > 0$.
- the operator $-L$ is uniformly elliptic, that is, there exists $\alpha_0 > 0$ such that

$$\sum_{k=1}^{d}\sum_{j=1}^{d} a_{jk}\xi_j\ \xi_k \geq \alpha_0|\xi|^2, \quad 0 \neq \xi \in \mathbb{R}^d.$$

In order to put into a first order system, set $v = u_t$ and rewrite (4.44) as a system:

$$\begin{aligned} u_t &= v, \quad v_t = Lu \quad \text{in } \Omega \times (0,\infty),\\ u &= 0 \quad \text{on } \partial\Omega \times (0,\infty),\\ u(x,0) &= g, \quad v(x,0) = h \quad \text{in } \Omega. \end{aligned}$$

Note that the following coercivity condition is satisfied: there exists a positive constant α_0 such that

$$(-L\phi,\phi) \geq \alpha_0\|\phi\|^2_{H^1_0(\Omega)} \quad \text{for all} \phi \in H^1_0(\Omega). \tag{4.48}$$

Now define X as a product space:

$$X = H^1_0(\Omega) \times L^2(\Omega)$$

with norm $\|(\phi,\psi)\| = \left(a(\phi,\phi) + \|v\|^2\right)^{1/2}$, where $a(\cdot,\cdot)$ is a bilinear form associated with the operator $-A$ given by

$$(-L\phi,\chi) := a(\phi,\chi) =: \sum_{j,k=1}^{d} \int_\Omega a_{jk}\frac{\partial\phi}{\partial x_k}\,\frac{\partial\chi}{\partial x_j}\,dx + \int_\Omega a_0\phi\,\chi\,dx.$$

Note that $t \longrightarrow (u(t),v(t)) \in X$ and we define operator A on the product space X as

$$A(u,v) = (v,-Lu) \tag{4.49}$$

with the domain of A is given by

$$D(A) = H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega).$$

It is easy to check that $D(A)$ is dense in X. It is left to the reader to verify that A is closed. Note that for $(u,v) \in D(A)$

$$(A(u,v),(u,v)) = a(v,u) + (-Lu,v) = a(v,u) - a(u,v) = 0$$

as $a(\cdot,\cdot)$ is symmetric and this implies that A is dissipative. Now for $\lambda > 0$, it remains to show that the Range of $(\lambda I - A)$ is X, that is, for any $(f_1,f_2) \in X$, the operator equation:

$$\lambda(u,v) - A(u,v) = (f_1,f_2)$$

has a unique solution $(u,v) \in D(A)$. Equivalently, the following two equations:

$$\lambda u - v = f_1 \quad \text{and} \quad \lambda v + Lu = f_2 \tag{4.50}$$

have a pair of solution $(u,v) \in D(A)$. On adding these two equations, it follows that

$$\lambda^2 u + Lu = \lambda f_1 + f_2. \tag{4.51}$$

Since $\lambda_1 f_1 + f_2 \in L^2(\Omega)$ and $\lambda^2 > 0$, we obtain from Lax-Milgram Lemma and elliptic regularity theory that, there exists a unique solution $u \in H^2(\Omega) \cap H^1_0(\Omega)$ to the problem (4.51). Since from (4.50), we obtain: $v = u - \lambda f_1 \in H^1_0(\Omega)$. Thus, we have shown that (4.50) has a unique solution $(u,v) \in D(A)$, for $(f_1,f_2) \in X$ which implies that the Range of $(\lambda I - A)$ is X. Now an application of the Lumer-Phillips theorem yields the existence of C^0 semigroup $E(t)$ of contraction.

References

[1] L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, Vol. 19, AMS, Providence, Rhode Island, 1998 (Reprinted 2002).

[2] S. Kesavan, *Topics in Functional Analysis and Applications*, New Age International(P) Limited, New Delhi, 1989 (Reprint 2003).

[3] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, Applied Math. Sciences, Vol. 44, 1983.