Chapter 4

Evolution equations

4.1 Introduction

The purpose of this chapter is to introduce some existence and regularity results for linear evolution equations. We consider equations of the form

$$z' = Az + f, \qquad z(0) = z_0. \tag{4.1.1}$$

In this setting A is an unbounded operator in a reflexive Banach space Z, with domain $D(A)$ dense in Z. We suppose that A is the infinitesimal generator of a strongly continuous semigroup on Z. This semigroup will be denoted by $(e^{tA})_{t\geq 0}$. In section 4.2, we study the weak solutions to equation (4.1.1) in $L^p(0,T;Z)$. For application to boundary control problems, we have to extend the notion of solutions to the case where $f \in L^p(0,T;(D(A^*))')$. In that case we study the solutions in $L^p(0,T;(D(A^*))')$ (see section 4.3). Before studying equation (4.1.1), let us now recall the Hille-Yosida theorem, which is very useful in applications.

Theorem 4.1.1 *([18, Chapter 1, Theorem 3.1], [8, Theorem 4.4.3]) An unbounded operator A with domain $D(A)$ in a Banach space Z is the infinitesimal generator of a strongly continuous semigroup of contractions if and only if the two following conditions hold:*
(i) A is a closed operator and $\overline{D(A)} = Z$,
(ii) for all $\lambda > 0$, $(\lambda I - A)$ is a bijective operator from $D(A)$ onto Z, $(\lambda I - A)^{-1}$ is a bounded operator on Z, and

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}(Z)} \leq \frac{1}{\lambda}.$$

Theorem 4.1.2 *Let $(e^{tA})_{t\geq 0}$ be a strongly continuous semigroup in Z with generator A. Then there exists $M \geq 1$ and $\omega \in \mathbb{R}$ such that*

$$\|e^{tA}\|_{\mathcal{L}(Z)} \leq Me^{\omega t} \qquad \text{for all } t \geq 0.$$

For all $c \in \mathbb{R}$, $A - cI$ is the infinitesimal generator of a strongly continuous semigroup on Z, denoted by $(e^{t(A-cI)})_{t\geq 0}$, which satisfies

$$\|e^{t(A-cI)}\|_{\mathcal{L}(Z)} \leq Me^{(\omega-c)t} \qquad \text{for all } t \geq 0.$$

The first part of the theorem can be found in [2, Chapter 1, Corollary 2.1], or in [18, Chapter 1, Theorem 2.2]. The second statement follows from that $e^{t(A-cI)} = e^{-ct}e^{tA}$.

4.2 Weak solutions in $L^p(0, T; Z)$

We recall the notion of weak solution to equation

$$z' = Az + f, \qquad z(0) = z_0, \tag{4.2.2}$$

where $z_0 \in Z$ and $f \in L^p(0, T; Z)$, with $1 \leq p < \infty$.

The adjoint operator of A is an unbounded operator in Z' defined by

$$D(A^*) = \{\zeta \in Z' \mid |\langle \zeta, Az \rangle| \leq c\|z\|_Z \text{ for all } z \in D(A)\}$$

and

$$\langle A^*\zeta, z\rangle = \langle \zeta, Az\rangle \quad \text{for all } \zeta \in D(A^*) \text{ and all } z \in D(A).$$

We know that the domain of A^* is dense in Z'.

Definition 4.2.1 *A function $z \in L^p(0, T; Z)$, with $1 \leq p < \infty$, is a weak solution to equation (4.2.2) if for every $\zeta \in D(A^*)$, $\langle z(\cdot), \zeta\rangle$ belongs to $W^{1,p}(0, T)$ and*

$$\frac{d}{dt}\langle z(t), \zeta\rangle = \langle z(t), A^*\zeta\rangle + \langle f(t), \zeta\rangle \quad \text{in }]0, T[, \qquad \langle z(0), \zeta\rangle = \langle z_0, \zeta\rangle.$$

Theorem 4.2.1 ([2, Chapter 1, Proposition 3.2]) *For every $z_0 \in Z$ and every $f \in L^p(0, T; Z)$, with $1 \leq p < \infty$, equation (4.2.2) admits a unique solution $z(f, z_0) \in L^p(0, T; Z)$, this solution belongs to $C([0, T]; Z)$ and is defined by*

$$z(t) = e^{tA}z_0 + \int_0^t e^{(t-s)A} f(s)\, ds.$$

The mapping $(f, z_0) \mapsto z(f, z_0)$ is linear and continuous from $L^p(0, T; Z) \times Z$ into $C([0, T]; Z)$.

The following regularity result is very useful.

Theorem 4.2.2 ([2, Chapter 1, Proposition 3.3]) *If $f \in C^1([0, T]; Z)$ and $z_0 \in D(A)$, then the solution z to equation (4.2.2) belongs to $C([0, T]; D(A)) \cap C^1([0, T]; Z)$.*

The adjoint equation for control problems associated with equation (4.2.2) will be of the form

$$-p' = A^*p + g, \qquad p(T) = p_T. \tag{4.2.3}$$

This equation can be studied with the following theorem.

Theorem 4.2.3 ([18, Chapter 1, Corollary 10.6]) *The family of operator $((e^{tA})^*)_{t\geq 0}$ is a strongly continuous semigroup on Z' with generator A^*. Since $e^{tA^*} = (e^{tA})^*$, $(e^{tA^*})_{t\geq 0}$ is called the adjoint semigroup of $(e^{tA})_{t\geq 0}$.*

Due to this theorem and to Theorem 4.2.1, with a change of time variable, it can be proved that if $p_T \in Z'$ and if $g \in L^p(0, T; Z')$, then equation (4.2.3) admits a unique weak solution which is defined by

$$p(t) = e^{(T-t)A^*}p_T + \int_t^T e^{(s-t)A^*} g(s)\, ds.$$

4.3 Weak solutions in $L^p(0,T;(D(A^*))')$

When the data of equation (4.2.2) are not regular, it is possible to extend the notion of solution by using duality arguments. It is the main objective of this section. For simplicity we suppose that Z is a Hilbert space (the results can be extended to the case where Z is a reflexive Banach space).

The imbeddings

$$D(A) \hookrightarrow Z \qquad \text{and} \qquad D(A^*) \hookrightarrow Z'$$

are continuous and with dense range. Thus we have

$$D(A) \hookrightarrow Z \hookrightarrow (D(A^*))'.$$

Since the operator $(A, D(A))$ is the infinitesimal generator of a strongly continuous semigroup on Z, from Theorem 4.2.3 it follows that $(A^*, D(A^*))$ is the infinitesimal generator of a semigroup on Z'. Let us denote by $(S^*(t))_{t\geq 0}$ this semigroup.

Recall that the operator $(A_1^*, D(A_1^*))$ defined by

$$D(A_1^*) = D((A^*)^2), \qquad A_1^* z = A^* z \quad \text{for all } z \in D(A_1^*),$$

is the infinitesimal generator of a semigroup on $D(A^*)$ and that this semigroup $(S_1^*(t))_{t\geq 0}$ obeys $S_1^*(t)z = S^*(t)z$ for all $z \in D(A^*)$.

From Theorem 4.2.3 we deduce that $((S_1^*)^*(t))_{t\geq 0}$ is the semigroup on $(D(A^*))'$ generated by $(A_1^*)^*$. We are going to show that $(S_1^*)^*(t)$ is the continuous extension of $S(t)$ to $(D(A^*))'$. More precisely we have the following

Theorem 4.3.1 *The adjoint of the unbounded operator $(A_1^*, D(A_1^*))$ in $D(A^*)$, is the unbounded operator $((A_1^*)^*, D((A_1^*)^*))$ on $(D(A^*))'$ defined by*

$$D((A_1^*)^*) = Z, \qquad \langle (A_1^*)^* z, y\rangle = \langle z, A_1^* y\rangle \qquad \text{for all } z \in Z \quad \text{and all } y \in D(A_1^*).$$

Moreover, $(A_1^*)^* z = Az$ for all $z \in D(A)$. The semigroup $((S_1^*)^*(t))_{t\geq 0}$ is the semigroup on $(D(A^*))'$ generated by $(A_1^*)^*$ and

$$(S_1^*)^*(t)z = S(t)z \qquad \text{for all } z \in Z \quad \text{and all } t \geq 0.$$

Proof. Let us show that $D((A_1^*)^*) = Z$. For all $z \in Z$ and all $y \in D(A_1^*)$, we have

$$|\langle z, A_1^* y\rangle_{(D(A^*))', D(A^*)}| = |\langle z, A_1^* y\rangle_{Z,Z'}| \leq \|z\|_Z \|y\|_{D(A^*)}.$$

Consequently

$$Z \subset D((A_1^*)^*). \tag{4.3.4}$$

Let us show the reverse inclusion. Let $z \in Z$ with $z \neq 0$, and let $y_z \in Z'$ be such that

$$\|z\|_Z = \sup_{y\in Z'} \frac{\langle z, y\rangle_{Z,Z'}}{\|y\|_{Z'}} = \frac{\langle z, y_z\rangle_{Z,Z'}}{\|y_z\|_{Z'}}.$$

We have

$$\|z\|_Z = \frac{\langle z, (I - A_1^*)(I - A_1^*)^{-1} y_z\rangle_{Z,Z'}}{\|y_z\|_{Z'}} = \frac{\langle z, (I - A_1^*)\zeta_z\rangle_{Z,Z'}}{\|\zeta_z\|_{D(A^*)}}$$

with $\zeta_z = (I - A_1^*)^{-1} y_z$. We can take

$$\zeta \longmapsto \|(I - A_1^*)^{-1}\zeta\|_{Z'}$$

as a norm on $D(A^*)$. For such a choice $(I - A_1^*)^{-1}$ is an isometry from Z' to $(D(A^*))'$. Thus

$$\sup_{\zeta\in D(A^*)} \frac{\langle z, (I - A_1^*)\zeta\rangle_{Z,Z'}}{\|\zeta\|_{D(A^*)}} = \sup_{y\in Z'} \frac{\langle z, y\rangle_{Z,Z'}}{\|y\|_{Z'}}.$$

Since

$$\|z\|_{D((A_1^*)^*)} = \sup_{\zeta\in D(A^*)} \frac{\langle z, (I - A_1^*)\zeta\rangle_{Z,Z'}}{\|\zeta\|_{D(A^*)}},$$

one has

$$\|z\|_{D((A_1^*)^*)} \le \|z\|_Z. \tag{4.3.5}$$

The equality $D((A_1^*)^*) = Z$ follows from (4.3.4) and (4.3.5).

For all $z \in D(A)$, and all $y \in D(A_1^*)$, we have

$$\langle (A_1^*)^* z, y\rangle = \langle z, A_1^* y\rangle = \langle z, A^* y\rangle = \langle Az, y\rangle.$$

Thus, $(A_1^*)^* z = Az$ for all $z \in D(A)$.

From Theorem 4.2.3 we deduce that $((S_1^*)^*(t))_{t\ge 0}$ is the semigroup on $(D(A^*))'$ generated by $(A_1^*)^*$. To prove that $(S_1^*)^*(t)z = S(t)z$ for all $z \in Z$ and all $t \ge 0$, it is sufficient to observe that

$$\langle (S_1^*)^*(t)z, y\rangle = \langle z, S_1^*(t)y\rangle = \langle z, S^*(t)y\rangle = \langle S(t)z, y\rangle,$$

for all $z \in Z$, all $y \in D(A^*)$, and all $t \ge 0$.

Remark. Therefore we can extend the notion of solution for the equation (4.2.2) in the case where $x_0 \in (D(A^*))'$ and $f \in L^p(0, T; (D(A^*))')$, by considering the equation

$$z'(t) = (A_1^*)^* z(t) + f(t) \qquad \text{dans } (0, T), \qquad z(0) = z_0. \tag{4.3.6}$$

It is a usual abuse of notation to replace A_1^* by A^* and to write equation (4.3.6) in the form (cf [2, page 160])

$$z'(t) = (A^*)^* z(t) + f(t) \qquad \text{dans } (0, T), \qquad z(0) = z_0. \tag{4.3.7}$$

Since $(A_1^*)^*$ is an extension of the operator A, sometimes equations (4.3.6) or (4.3.7) are written in the form (4.2.2) even if $z_0 \in (D(A^*))'$ and $f \in L^p(0, T; (D(A^*))')$.

Theorem 4.3.2 *For every $z_0 \in (D(A^*))'$ and every $f \in L^p(0, T; (D(A^*))')$, with $1 \le p < \infty$, equation (4.3.6) admits a unique solution $z(f, z_0) \in L^p(0, T; (D(A^*))')$, this solution belongs to $C([0, T]; (D(A^*))')$ and is defined by*

$$z(t) = e^{t(A_1^*)^*} z_0 + \int_0^t e^{(t-s)(A_1^*)^*} f(s)\, ds.$$

The mapping $(f, z_0) \mapsto z(f, z_0)$ is linear and continuous from $L^p(0, T; (D(A^*))') \times (D(A^*))'$ into $C([0, T]; (D(A^*))')$.

Proof. The theorem is a direct consequence of Theorems 4.3.1 and 4.2.1. ∎

For simplicity in the notation, we often write e^{tA} in place of $e^{t(A_1^*)^*}$, or A in place of $(A_1^*)^*$.

We often establish identities by using density arguments. The following regularity result will be useful to establish properties for weak solutions to equation (4.3.4).

Theorem 4.3.3 *If f belongs to $H^1(0,T;(D(A^*))')$ and z_0 belongs to Z, then the solution $z(f,z_0)$ to equation (4.3.4) belongs to $C^1([0,T];(D(A^*))')\cap C([0,T];Z)$.*

Proof. See [2, Chapter 3, Theorem 1.1].

4.4 Analytic semigroups

Let $(A,D(A))$ be the infinitesimal generator of a strongly continuous semigroup on a Hilbert space Z. The resolvent set $\rho(A)$ is the set of all complex numbers λ such that the operator $(\lambda I - A)\in\mathcal{L}(D(A),Z)$ has a bounded inverse in Z. Since Z is a Hilbert space, and A is a closed operator (because A is the infinitesimal generator of a strongly continuous semigroup), we have the following characterization of $\rho(A)$:

$\lambda\in\rho(A)$ if and only if $R(\lambda,A)=(\lambda I-A)^{-1}$ exists and $\mathrm{Im}(\lambda I-A)=Z$.

The resolvent set of A always contains a real half-line $[a,\infty)$ (see [2, Chapter 1, Proposition 2.2 and Corollary 2.2]).

4.4.1 Fractional powers of infinitesimal generators

We follows the lines of [5, Section 7.4]. Let $(e^{tA})_{t\geq 0}$ be a strongly continuous semigroup on Z with infinitesimal generator A satisfying

$$\|e^{tA}\|_{\mathcal{L}(Z)}\leq Me^{-ct}\qquad\text{for all } t\geq 0, \tag{4.4.8}$$

with $c>0$. We can define fractional powers of $(-A)$ by

$$(-A)^{-\alpha}z=\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{tA}z\,dt$$

for some $\alpha>0$ and all $z\in Z$. The operator $(-A)^{-\alpha}$ obviously belongs to $\mathcal{L}(Z)$. For $0\leq\alpha\leq 1$, we set

$$(-A)^{\alpha}=(-A)(-A)^{\alpha-1}.$$

The domain of $(-A)^\alpha=(-A)(-A)^{\alpha-1}$ is defined by $D((-A)^\alpha)=\{z\in Z\mid(-A)^{\alpha-1}z\in D(A)\}$.

4.4.2 Analytic semigroups

Different equivalent definitions of an analytic semigroup can be given.

Definition 4.4.1 *Let $(e^{tA})_{t\geq 0}$ be a strongly continuous semigroup on Z, with infinitesimal generator A. The semigroup $(e^{tA})_{t\geq 0}$ is analytic if there exists a sector*

$$\Sigma_{a,\frac{\pi}{2}+\delta} = \{\lambda \in \mathbb{C} \mid |arg(\lambda - a)| < \frac{\pi}{2} + \delta\}$$

with $0 < \delta < \frac{\pi}{2}$, such that $\Sigma_{a,\frac{\pi}{2}+\delta} \subset \rho(A)$, and

$$\|(\lambda I - A)^{-1}\| \leq \frac{M}{|\lambda - a|} \qquad \text{for all } \lambda \in \Sigma_{a,\frac{\pi}{2}+\delta}.$$

It can be proved that the semigroup $(e^{tA})_{t\geq 0}$ satisfies the conditions of definition 4.4.1 if and only if $(e^{tA})_{t\geq 0}$ can be extended to a function $\lambda \mapsto e^{\lambda A}$, where $e^{\lambda A} \in \mathcal{L}(Z)$, analytic in the sector

$$\Sigma_{a,\delta} = \{\lambda \in \mathbb{C} \mid |\arg(\lambda - a)| < \delta\},$$

and strongly continuous in

$$\{\lambda \in \mathbb{C} \mid |\arg(\lambda - a)| \leq \delta\}.$$

Such a result can be find in a slightly different form in [2, Chapter 1, Theorem 2.1]. A theorem very useful for studying regularity of solutions to evolution equations is stated below.

Theorem 4.4.1 (*[18, Chapter 2, Theorem 6.13]*) *Let $(e^{tA})_{t\geq 0}$ be a continuous semigroup with infinitesimal generator A. Suppose that (4.4.8) is satisfied for some $c > 0$. Then $e^{tA}Z \subset D((-A)^{\alpha})$, $(-A)^{\alpha}e^{tA} \in \mathcal{L}(Z)$ for all $t > 0$, and, for all $0 \leq \alpha \leq 1$, there exists $k > 0$ and $C(\alpha)$ such that*

$$\|(-A)^{\alpha}e^{tA}\|_{\mathcal{L}(Z)} \leq C(\alpha)t^{-\alpha}e^{-kt} \qquad \text{for all } t \geq 0. \tag{4.4.9}$$

A very simple criterion of analitycity is known in the case of real Hilbert spaces.

Theorem 4.4.2 (*[2, Chapter 1, Proposition 2.11]*) *If A is a selfadjoint operator on a real Hilbert space Z, and if*

$$(Az, z) \leq 0 \qquad \text{for all } z \in D(A),$$

then A generates an analytic semigroup of contractions on Z.

Chapter 5

Control of the heat equation

5.1 Introduction

We begin with distributed controls (section 5.2). Solutions of the heat equation are defined via the semigroup theory, but we explain how we can recover regularity results in $W(0,T;H_0^1(\Omega),H^{-1}(\Omega))$ (Theorem 5.2.3). Since we study optimal control problems of evolution equations for the first time, we carefully explain how we can calculate the gradient, with respect to the control variable, of a functional depending of the state variable via the adjoint state method. The case of Neumann boundary controls is studied in section 5.3. Estimates in $W(0,T;H^1(\Omega),(H^1(\Omega))')$ are obtained by an approximation process, using the Neumann operator (see the proof of Theorem 5.3.6). Section 5.4 deals with Dirichlet boundary controls. In that case the solutions do not belong to $C([0,T];L^2(\Omega))$, but only to $C([0,T];H^{-1}(\Omega))$. We carefully study control problems for functionals involving observations in $C([0,T];H^{-1}(\Omega))$ (see section 5.4.2).

We only study problems without control constraints. But the extension of existence results and optimality conditions to problems with control constraints is straightforward.

5.2 Distributed control

Let Ω be a bounded domain in $\mathbb{R}^N$, with a boundary Γ of class C^2. Let $T>0$, set $Q=\Omega\times(0,T)$ and $\Sigma=\Gamma\times(0,T)$. We consider the heat equation with a distributed control

$$\frac{\partial z}{\partial t}-\Delta z=f+\chi_\omega u \quad \text{in } Q, \quad z=0 \quad \text{on } \Sigma, \quad z(x,0)=z_0 \quad \text{in } \Omega. \tag{5.2.1}$$

The function f is a given source of temperature, χ_ω is the characteristic function of ω, ω is an open subset of Ω, and the function u is a control variable. We consider the control problem

$$(P_1) \qquad \inf\{J_1(z,u)\mid (z,u)\in C([0,T];L^2(\Omega))\times L^2(0,T;L^2(\omega)),\ (z,u) \text{ satisfies } (5.2.1)\},$$

where

$$J_1(z,u)=\frac{1}{2}\int_Q(z-z_d)^2+\frac{1}{2}\int_\Omega(z(T)-z_d(T))^2+\frac{\beta}{2}\int_Q\chi_\omega u^2,$$

and $\beta>0$. In this section, we assume that $f\in L^2(Q)$ and that $z_d\in C([0,T];L^2(\Omega))$.

Before studying the above control problem, we first recall some results useful for the equation

$$\frac{\partial z}{\partial t} - \Delta z = \phi \quad \text{in } Q, \quad z = 0 \quad \text{on } \Sigma, \quad z(x,0) = z_0(x) \quad \text{in } \Omega. \tag{5.2.2}$$

Theorem 5.2.1 *Set $Z = L^2(\Omega)$, $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $Au = \Delta u$. The operator $(A, D(A))$ is the infinitesimal generator of a strongly continuous semigroup of contractions on $L^2(\Omega)$.*

Proof. The proof relies on the Hille-Yosida theorem and on regularity properties for solutions to the Laplace equation.

(i) Prove that A is a closed operator. Let $(z_n)_n$ be a sequence in $D(A)$ converging to some z in $L^2(\Omega)$. Suppose that $(\Delta z_n)_n$ converges to some f in $L^2(\Omega)$. We necessarily have $\Delta z = f$ in the sense of distributions in Ω. Due to Theorem 3.2.1, we have $\|z_n - z_m\|_{H^2(\Omega)} \le C\|\Delta z_n - \Delta z_m\|_{L^2(\Omega)}$. This means that $(z_n)_n$ is a Cauchy sequence in $H^2(\Omega)$. Hence $z \in H^2(\Omega) \cap H_0^1(\Omega)$. The first condition of Theorem 4.1.1 is satisfied.

(ii) Let $\lambda > 0$ and $f \in L^2(\Omega)$. It is clear that $(\lambda I - A)$ is invertible in $L^2(\Omega)$, and $(\lambda I - A)^{-1} f$ is the solution z to the equation

$$\lambda z - \Delta z = f \qquad \text{in } \Omega, \qquad z = 0 \qquad \text{on } \Gamma.$$

We know that $z \in H^2(\Omega) \cap H_0^1(\Omega)$ and

$$\lambda \int_\Omega z^2 + \int_\Omega |\nabla z|^2 = \int_\Omega fz.$$

Thus we have

$$\|z\|_{L^2(\Omega)} \le \frac{1}{\lambda}\|f\|_{L^2(\Omega)},$$

and the proof is complete. ∎

Equation (5.2.2) may be rewritten in the form of an evolution equation:

$$z' - Az = \phi \quad \text{in }]0,T[, \qquad z(0) = z_0. \tag{5.2.3}$$

We can easily verify that $D(A^*) = D(A)$ and $A^* = A$, that is A is selfadjoint. Recall that $z \in L^2(0,T;L^2(\Omega))$ is a weak solution to equation (5.2.3) if for all $\zeta \in H^2(\Omega) \cap H_0^1(\Omega)$ the mapping $t \mapsto \langle z(t), \zeta\rangle$ belongs to $H^1(0,T)$, $\langle z(0), \zeta\rangle = \langle z_0, \zeta\rangle$, and

$$\frac{d}{dt}\langle z(t), \zeta\rangle = \langle z(t), A\zeta\rangle + \langle \phi, \zeta\rangle.$$

Theorem 5.2.2 *(i) For every $\phi \in L^2(Q)$ and every $z_0 \in L^2(\Omega)$, equation (5.2.2) admits a unique weak solution $z(\phi, z_0)$ in $L^2(0,T;L^2(\Omega))$, moreover the operator is linear and continuous from $L^2(Q) \times L^2(\Omega)$ into $W(0,T;H_0^1(\Omega), H^{-1}(\Omega))$.*

(ii) The operator is also continuous from $L^2(Q) \times H_0^1(\Omega)$ into $L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$.

Comments. Recall that

$$W(0,T;H_0^1(\Omega), H^{-1}(\Omega)) = \Big\{z \in L^2(0,T;H_0^1(\Omega)) \mid \frac{dz}{dt} \in L^2(0,T;H^{-1}(\Omega))\Big\}.$$

We say that $\frac{dz}{dt} \in L^2(0,T;H^{-1}(\Omega))$ if

$$\|\frac{d}{dt}\langle z(t),\zeta\rangle\|_{L^2(0,T)} \le C\|\zeta\|_{H^1_0(\Omega)}, \qquad \text{for all } \ \zeta \in H^1_0(\Omega).$$

Proof of Theorem 5.2.2.
(i) Due to Theorem 5.2.1 and Theorem 4.2.1, we can prove that the operator $(\phi, z_0) \mapsto z(\phi, z_0)$ is continuous from $L^2(Q) \times L^2(\Omega)$ into $C([0,T];L^2(\Omega))$. To prove that the solution z belongs to $W(0,T;H^1_0(\Omega),H^{-1}(\Omega))$, we can use a density argument. Suppose that $\phi \in C^1([0,T];Z)$ and that $z_0 \in D(A)$. Then z belongs to $C([0,T];D(A)) \cap C^1([0,T];Z)$ (Theorem 4.2.2). In that case we can multiply equation (5.2.2) by z, and with integration by parts and a Green formula, we obtain

$$\begin{aligned}\int_\Omega |z(T)|^2 + 2\int_0^T\int_\Omega |\nabla z|^2 &\le 2\int_0^T\int_\Omega \phi z + \int_\Omega |z_0|^2 \\ &\le 2\|\phi\|_{L^2(Q)}\|z\|_{L^2(Q)} + \|z_0\|^2_{L^2(\Omega)}.\end{aligned}$$

With Poincaré's inequality $\|z\|_{L^2(\Omega)} \le C_p\|\nabla z\|_{L^2(\Omega)}$, and Young' inequality we deduce

$$\int_0^T\int_\Omega |\nabla z|^2 \le C_p\|\phi\|^2_{L^2(Q)} + \|z_0\|^2_{L^2(\Omega)}.$$

Therefore the operator $(\phi, z_0) \mapsto z(\phi, z_0)$ is continuous from $L^2(Q)\times L^2(\Omega)$ into $L^2(0,T;H^1_0(\Omega))$. Next, by using the equation and the regularity $z \in L^2(0,T;H^1_0(\Omega))$, we get

$$\frac{d}{dt}\langle z(t),\zeta\rangle = \langle z(t), A\zeta\rangle + \langle \phi,\zeta\rangle = -\int_\Omega \nabla z\nabla\zeta + \int_\Omega \phi\zeta.$$

From which it follows that

$$\begin{aligned}\|\frac{d}{dt}\langle z(t),\zeta\rangle\|_{L^2(0,T)} &\le \|z\|_{L^2(0,T;H^1_0(\Omega))}\|\zeta\|_{H^1_0(\Omega)} + \|\phi\|_{L^2(\Omega)}\|\zeta\|_{L^2(\Omega)} \\ &\le \max(C_p,1)\Big(\|z\|_{L^2(0,T;H^1_0(\Omega))} + \|\phi\|_{L^2(\Omega)}\Big)\|\zeta\|_{H^1_0(\Omega)},\end{aligned}$$

for all $\zeta \in H^1_0(\Omega)$. Thus $\frac{dz}{dt}$ belongs to $L^2(0,T;H^{-1}(\Omega))$. The first part of the Theorem is proved.

(ii) The second regularity result is proved in [13], [7]. ∎

Since the solution $z(f,u,z_0)$ to equation (5.2.1) belongs to $C([0,T];L^2(\Omega))$ (when $u \in L^2(0,T;L^2(\omega))$), $J_1(z(f,u,z_0),u)$ is well defined and is finite for any $u \in L^2(0,T;L^2(\omega))$. We first assume that (P_1) admits a unique solution (see Theorem 7.3.1, see also exercise 5.5.1). We set $F_1(u) = J_1(z(f,u,z_0),u)$, and, as in the case of optimal control for elliptic equations, the optimal solution $(z(f,\bar u,z_0),\bar u)$ to problem (P_1) is characterized by the equation $F_1'(\bar u) = 0$. To compute the gradient of F_1 we have to consider adjoint equations of the form

$$-\frac{\partial p}{\partial t} - \Delta p = g \quad \text{in } Q, \quad p = 0 \quad \text{on } \Sigma, \quad p(x,T) = p_T \quad \text{in } \Omega, \tag{5.2.4}$$

with $g \in L^2(Q)$ and $p_T \in L^2(\Omega)$. It is well known that the backward heat equation is not well posed. Due to the condition $p(x,T) = p_T$ equation (5.2.4) is a terminal value problem, which must be integrated backward in time. But equation (5.2.4) is not a backward heat equation because we have $-\frac{\partial p}{\partial t} - \Delta p = g$ and not $\frac{\partial p}{\partial t} - \Delta p = g$ (as in the case of the backward heat equation). Let us explain why the equation is well posed. If p is a solution of (5.2.4) and if we set $w(t) = p(T-t)$, we can check, at least formally, that w is the solution of

$$\frac{\partial w}{\partial t} - \Delta w = g(x, T-t) \quad \text{in } Q, \quad w = 0 \quad \text{on } \Sigma, \quad w(x,0) = p_T \quad \text{in } \Omega. \tag{5.2.5}$$

Since equation (5.2.5) is well posed, equation (5.2.4) is also well posed even if (5.2.4) is a *terminal value problem*. In particular equation (5.2.4) admits a unique weak solution in $L^2(0,T;L^2(\Omega))$, and this solution belongs to $W(0,T;H_0^1(\Omega),H^{-1}(\Omega))$. To obtain the expression of the gradient of F_1 we need a Green formula which is stated below.

Theorem 5.2.3 *Suppose that $\phi \in L^2(Q)$, $g \in L^2(Q)$, and $p_T \in L^2(\Omega)$. Then the solution z of equation*

$$\frac{\partial z}{\partial t} - \Delta z = \phi \quad \textit{in } Q, \quad z = 0 \quad \textit{on } \Sigma, \quad z(x,0) = 0 \quad \textit{in } \Omega,$$

and the solution p of (5.2.4) satisfy the following formula

$$\int_Q \phi\, p = \int_Q z\, g + \int_\Omega z(T) p_T. \tag{5.2.6}$$

Proof. If $p_T \in H_0^1(\Omega)$, due to Theorem 5.2.2, z and p belong to $L^2(0,T;D(A)))\cap H^1(0,T;L^2(\Omega))$. In that case, with the Green formula we have

$$\int_\Omega -\Delta z(t) p(t)\, dx = \int_\Omega -\Delta p(t) z(t)\, dx$$

for almost every $t \in [0,T]$, and

$$\int_0^T \int_\Omega \frac{\partial z}{\partial t} p = -\int_0^T \int_\Omega \frac{\partial p}{\partial t} z + \int_\Omega z(T) p_T.$$

Thus formula (5.2.6) is established in the case when $p_T \in H_0^1(\Omega)$ (Theorem 5.2.2 (ii)). If $(p_{Tn})_n$ is a sequence in $H_0^1(\Omega)$ converging to p_T in $L^2(\Omega)$, due to Theorem 5.2.2, $(p_n)_n$, where p_n is the solution to equation (5.2.4) corresponding to p_{Tn}, converges to p in $W(0,T;H_0^1(\Omega),H^{-1}(\Omega))$ when n tends to infinity. Thus, in the case when $p_T \in L^2(\Omega)$, formula (5.2.6) can be deduced by passing to the limit in the formula satisfied by p_n.

The gradient of F_1. Let $(z(f,\bar{u},z_0),\bar{u}) = (\bar{z},\bar{u})$ be the solution to problem (P_1). By a direct calculation we obtain

$$F_1(\bar{u}+\lambda u) - F_1(\bar{u}) = \frac{1}{2}\int_Q (z_\lambda - \bar{z})(z_\lambda + \bar{z} - 2z_d)$$

$$+\frac{1}{2}\int_\Omega (z_\lambda(T) - \bar{z}(T))(z_\lambda(T) + \bar{z}(T) - 2z_d(T)) + \frac{\beta}{2}\int_0^T \int_\omega (2\lambda u\bar{u} + \lambda^2 u^2),$$

where $z_\lambda = z(f, \bar{u} + \lambda u, z_0)$. The function $w_\lambda = z_\lambda - \bar{z}$ is the solution to the equation

$$\frac{\partial w}{\partial t} - \Delta w = \lambda \chi_\omega u \quad \text{in } Q, \quad z = 0 \quad \text{on } \Sigma, \quad w(x, 0) = 0 \quad \text{in } \Omega.$$

Due to Theorem 5.2.2 we have

$$\|w_\lambda\|_{W(0,T;H_0^1(\Omega),H^{-1}(\Omega))} \leq C|\lambda|\|u\|_{L^2(0,T;L^2(\omega))}.$$

Thus the sequence $(z_\lambda)_\lambda$ converges to $\bar{z}$ in $W(0, T; H_0^1(\Omega), H^{-1}(\Omega))$ when λ tends to zero. Set $w_u = \frac{1}{\lambda} w_\lambda$, the function w_u is the solution to the equation

$$\frac{\partial w}{\partial t} - \Delta w = \chi_\omega u \quad \text{in } Q, \quad z = 0 \quad \text{on } \Sigma, \quad w(x, 0) = 0 \quad \text{in } \Omega. \tag{5.2.7}$$

Dividing $F_1(\bar{u} + \lambda u) - F_1(\bar{u})$ by λ, and passing to the limit when λ tends to zero, we obtain:

$$F_1'(\bar{u})u = \int_Q (\bar{z} - z_d) w_u + \int_\Omega (\bar{z}(T) - z_d(T)) w_u(T) + \int_0^T \int_\omega \beta u \bar{u}.$$

To derive the expression of $F_1'(\bar{u})$ we introduce the adjoint equation

$$-\frac{\partial p}{\partial t} - \Delta p = \bar{z} - z_d \quad \text{in } Q, \quad p = 0 \quad \text{on } \Sigma, \quad p(x, T) = \bar{z}(T) - z_d(T) \quad \text{in } \Omega. \tag{5.2.8}$$

With formula (5.2.6) applied to p and w_u we have

$$\int_Q (\bar{z} - z_d) w_u + \int_\Omega (\bar{z}(T) - z_d(T)) w_u(T) = \int_0^T \int_\omega \chi_\omega u p.$$

Hence $F_1'(\bar{u}) = p|_{\omega \times (0,T)} + \beta \bar{u}$, where p is the solution to equation (5.2.8).

Theorem 5.2.4 *(i) If $(\bar{z}, \bar{u})$ is the solution to (P_1) then $\bar{u} = -\frac{1}{\beta} p|_{\omega \times (0,T)}$, where p is the solution to equation (5.2.8).*

(ii) Conversely, if a pair $(\tilde{z}, \tilde{p}) \in W(0, T; H_0^1(\Omega), H^{-1}(\Omega)) \times W(0, T; H_0^1(\Omega), H^{-1}(\Omega))$ obeys the system

$$\begin{aligned} &\frac{\partial z}{\partial t} - \Delta z = f - \frac{1}{\beta} \chi_\omega p \quad \textit{in } Q, \quad p = 0 \quad \textit{on } \Sigma, \quad z(x, 0) = \bar{z}_0 \quad \textit{in } \Omega, \\ &-\frac{\partial p}{\partial t} - \Delta p = \bar{z} - z_d \quad \textit{in } Q, \quad p = 0 \quad \textit{on } \Sigma, \quad p(x, 0) = \bar{z}(T) - z_d(T) \quad \textit{in } \Omega, \end{aligned} \tag{5.2.9}$$

then the pair $(\tilde{z}, -\frac{1}{\beta}\tilde{p})$ is the optimal solution to problem (P_1).

Proof. (i) The necessary optimality condition is already proved.

(ii) The sufficient optimality condition can be proved with Theorem 2.2.3. ∎

Comments. Before ending this section let us observe that equation (5.2.1) can be written in the form

$$z' = Az + f + Bu, \qquad z(0) = z_0,$$

where $B \in \mathcal{L}(L^2(\Gamma), L^2(\Omega))$ is defined by $Bu = \chi_\omega u$. Control problems governed by such evolutions equations are studied in Chapter 7.

5.3 Neumann boundary control

In this section, we study problems in which the control variable acts through a Neumann boundary condition

$$\frac{\partial z}{\partial t} - \Delta z = f \quad \text{in } Q, \quad \frac{\partial z}{\partial n} = u \quad \text{on } \Sigma, \quad z(x,0) = z_0 \quad \text{in } \Omega. \tag{5.3.10}$$

Theorem 5.3.1 *Set $Z = L^2(\Omega)$, $D(A) = \{z \in H^2(\Omega) \mid \frac{\partial z}{\partial n} = 0\}$, $Az = \Delta z$. The operator $(A, D(A))$ is the infinitesimal generator of a strongly continuous semigroup of contractions in $L^2(\Omega)$.*

Proof. The proof still relies on the Hille-Yosida theorem. It is very similar to the proof of Theorem 5.2.1 and is left to the reader. ∎

The operator $(A, D(A))$ is selfadjoint in Z. Equation

$$\frac{\partial z}{\partial t} - \Delta z = f \quad \text{in } Q, \quad \frac{\partial z}{\partial n} = 0 \quad \text{on } \Sigma, \quad z(x,0) = z_0 \quad \text{in } \Omega, \tag{5.3.11}$$

may be written in the form

$$z' = Az + f, \qquad z(0) = z_0. \tag{5.3.12}$$

A function $z \in L^2(0,T;L^2(\Omega))$ is a weak solution to equation (5.3.12) if for all $\zeta \in D(A)$ the mapping $t \mapsto \langle z(t), \zeta\rangle$ belongs to $H^1(0,T)$, $\langle z(0), \zeta\rangle = \langle z_0, \zeta\rangle$, and

$$\frac{d}{dt}\int_\Omega z(t)\zeta = \langle z, A\zeta\rangle + \langle f, \zeta\rangle = \int_\Omega z(t)\Delta\zeta + \int_\Omega f(t)\zeta.$$

Theorem 5.3.2 *For every $\phi \in L^2(Q)$ and every $z_0 \in L^2(\Omega)$, equation (5.3.11) admits a unique weak solution $z(\phi, z_0)$ in $L^2(0,T;L^2(\Omega))$, moreover the operator*

$$(\phi, z_0) \mapsto z(\phi, z_0)$$

is linear and continuous from $L^2(Q) \times L^2(\Omega)$ into $W(0,T;H^1(\Omega),(H^1(\Omega))')$.

Recall that

$$W(0,T;H^1(\Omega),(H^1(\Omega))') = \Big\{z \in L^2(0,T;H^1(\Omega)) \mid \frac{dz}{dt} \in L^2(0,T;(H^1(\Omega))')\Big\}.$$

Proof. The existence in $C([0,T];L^2(\Omega))$ follows from Theorem 5.3.1. The regularity in $W(0,T;H^1(\Omega),(H^1(\Omega))')$ can be proved as for Theorem 5.2.2. ∎

Similarly we would like to say that a function $z \in L^2(0,T;L^2(\Omega))$ is a weak solution to equation (5.3.10) if for all $\zeta \in D(A)$ the mapping $t \mapsto \langle z(t), \zeta\rangle$ belongs to $H^1(0,T)$, $\langle z(0), \zeta\rangle = \langle z_0, \zeta\rangle$, and

$$\frac{d}{dt}\int_\Omega z(t)\zeta = \int_\Omega z(t)\Delta\zeta + \int_\Omega f\zeta + \int_\Gamma u\zeta.$$

Unfortunately the mapping $\zeta \mapsto \int_\Gamma u\zeta$ is not an element of $L^2(0,T;L^2(\Omega))$, it only belongs to $L^2(0,T;(H^1(\Omega))')$. One way to study equation (5.3.10) consists in using $(A_1^*)^*$ (see Chapter 4), the extension of A to $(D(A^*))' = (D(A))'$ (A is selfadjoint). We can directly improve this result in the following way. We set $\widehat{Z} = (H^1(\Omega))'$. We endow $(H^1(\Omega))'$ with the dual norm of the H^1-norm. We can check that the corresponding inner product in $(H^1(\Omega))'$ is defined by

$$\Big(z,\zeta\Big)_{(H^1(\Omega))'} = \int_\Omega z(-\Delta+I)^{-1}\zeta = \int_\Omega (-\Delta+I)^{-1}z\,\zeta,$$

where $(-\Delta+I)^{-1}\zeta$ is the function $w\in H^1(\Omega)$ obeying

$$-\Delta w + w = \zeta \quad \text{in } \Omega, \qquad \frac{\partial w}{\partial n} = 0 \quad \text{on } \Gamma.$$

We define the unbounded operator $\widehat{A}$ in $(H^1(\Omega))'$ by $D(\widehat{A}) = H^1(\Omega)$, and

$$\langle \widehat{A}z,\zeta\rangle_{(H^1(\Omega))',H^1(\Omega)} = -\int_\Omega \nabla z\nabla\zeta = \Big(\widehat{A}z,\zeta\Big)_{(H^1(\Omega))'}.$$

Theorem 5.3.3 *The operator $(\widehat{A}, D(\widehat{A}))$ is the infinitesimal generator of a strongly continuous semigroup of contractions in $(H^1(\Omega))'$.*

Proof. The proof still relies on the Hille-Yosida theorem. It is more complicated than the previous ones. It is left to the reader. ∎

We write equation (5.3.10) in the form

$$z' = \widehat{A}z + f + \hat{u}, \qquad z(0) = z_0, \tag{5.3.13}$$

where $\hat{u} \in L^2(0,T;(H^1(\Omega))')$ is defined by $\hat{u} \mapsto \int_\Gamma u\zeta$ for all $\zeta\in H^1(\Omega)$. Due to Theorem 5.3.3 equation (5.3.13), or equivalently equation (5.3.10), admits a unique solution in $L^2(0,T;(H^1(\Omega))')$ and this solution belongs to $C([0,T];(H^1(\Omega))')$. To establish regularity properties of solutions to equation (5.3.10) we need to construct solutions by an approximation process.

Approximation by regular solutions.

Recall that the solution to equation

$$\Delta w - w = 0 \quad \text{in } \Omega, \quad \frac{\partial w}{\partial n} = v \quad \text{on } \Gamma, \tag{5.3.14}$$

satisfies the estimate

$$\|w\|_{H^2(\Omega)} \le C\|v\|_{H^{1/2}(\Gamma)}. \tag{5.3.15}$$

Let u be in $L^2(\Sigma)$ and let $(u_n)_n$ be a sequence in $C^1([0,T];H^{1/2}(\Gamma))$, converging to u in $L^2(\Sigma)$. Denote by $w_n(t)$ the solution to equation (5.3.14) corresponding to $v = u_n(t)$. With estimate (5.3.15) we can prove that w_n belongs to $C^1([0,T];H^2(\Omega))$ and that

$$\|w_n\|_{C^1([0,T];H^2(\Omega))} \le C\|u_n\|_{C^1([0,T];H^{1/2}(\Gamma))}.$$

Let z_n be the solution to equation (5.3.10) corresponding to (f,u_n,z_0). Then $y_n = z_n - w_n$ is the solution to

$$\frac{\partial y}{\partial t} - \Delta y = f - \frac{\partial w_n}{\partial t} + \Delta w_n \quad \text{in } Q, \quad \frac{\partial y}{\partial n} = 0 \quad \text{on } \Sigma, \quad y(x,0) = (z_0 - w_n(0))(x) \quad \text{in } \Omega.$$

Since $(z_0 - w_n(0)) \in L^2(\Omega)$ and $f - \frac{\partial w_n}{\partial t} - \Delta w_n$ belongs to $L^2(Q)$, y_n and z_n belong to $W(0,T;H^1(\Omega),(H^1(\Omega))')$. Thus, for every $t \in]0,T]$, we have

$$\int_\Omega |z_n(t)|^2 + 2\int_0^t\int_\Omega |\nabla z_n|^2 = 2\int_0^t\int_\Omega f z_n + 2\int_0^t\int_\Gamma u_n z_n + \int_\Omega |z_0|^2.$$

We first get

$$\|y\|^2_{C([0,T];L^2(\Omega))} + 2\|\nabla y\|^2_{L^2(0,T;L^2(\Omega))} \le 2\|f\|_{L^2(Q)}\|y\|_{L^2(Q)} + 2\|u\|_{L^2(\Sigma)}\|y\|_{L^2(\Sigma)} + \|y_0\|^2_{L^2(\Omega)}.$$

Thus with Young's inequality, we obtain

$$\|y\|_{C([0,T];L^2(\Omega))} + \|y\|_{L^2(0,T;H^1(\Omega))} \le C\Big(\|f\|_{L^2(Q)} + \|u\|_{L^2(\Sigma)} + \|y_0\|_{L^2(\Omega)}\Big).$$

In the same way, we can prove

$$\|z_n - z_m\|_{C([0,T];L^2(\Omega))} + \|z_n - z_m\|_{L^2(0,T;H^1(\Omega))} \le C\|u_n - u_m\|_{L^2(\Sigma)}.$$

Hence the sequence $(z_n)_n$ converges to some z in $C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$. Due to Theorem 5.3.3, we can also prove that the sequence $(z_n)_n$ converges to the solution of equation (5.3.10) in $C([0,T];L^2(\Omega))$. By using the same arguments as for Theorem 5.2.2, we can next prove an estimate in $W(0,T;H^1(\Omega),(H^1(\Omega))')$. Therefore we have established the following theorem.

Theorem 5.3.4 *For every $f \in L^2(Q)$, every $u \in L^2(\Sigma)$, and every $z_0 \in L^2(\Omega)$, equation (5.3.10) admits a unique weak solution $z(f,u,z_0)$ in $L^2(0,T;L^2(\Omega))$, moreover the operator*

$$(f,u,z_0) \mapsto z(f,u,z_0)$$

is linear and continuous from $L^2(Q)\times L^2(\Sigma)\times L^2(\Omega)$ into $W(0,T;H^1(\Omega),(H^1(\Omega))')$.

We now consider the control problem

$$(P_2) \qquad \inf\{J_2(z,u) \mid (z,u) \in C([0,T];L^2(\Omega)) \times L^2(0,T;L^2(\Gamma)),\ (z,u) \text{ satisfies (5.3.10)}\},$$

where

$$J_2(z,u) = \frac{1}{2}\int_Q (z-z_d)^2 + \frac{1}{2}\int_\Omega (z(T)-z_d(T))^2 + \frac{\beta}{2}\int_\Sigma u^2.$$

We assume that $f \in L^2(Q)$, $z_0 \in L^2(\Omega)$, and $z_d \in C([0,T];L^2(\Omega))$. Problem (P_2) admits a unique solution $(\bar z, \bar u)$ (see exercise 5.5.2). The adjoint equation for (P_2) is of the form

$$-\frac{\partial p}{\partial t} - \Delta p = g \quad \text{in } Q, \qquad \frac{\partial p}{\partial n} = 0 \quad \text{on } \Sigma, \qquad p(x,T) = p_T \quad \text{in } \Omega. \tag{5.3.16}$$

Theorem 5.3.5 *Suppose that $u \in L^2(\Sigma)$, $g \in L^2(Q)$, $p_T \in L^2(\Omega)$. Then the solution z of equation*

$$\frac{\partial z}{\partial t} - \Delta z = 0 \quad \text{in } Q, \qquad \frac{\partial z}{\partial n} = u \quad \text{on } \Sigma, \qquad z(0) = 0 \quad \text{in } \Omega,$$

and the solution p of (5.3.16) satisfy the following formula

$$\int_\Sigma up = \int_Q z\,g + \int_\Omega z(T)p_T. \tag{5.3.17}$$

Proof. We leave the reader adapt the proof of Theorem 5.2.3.

Theorem 5.3.6 *If $(\bar{z}, \bar{u})$ is the solution to (P_2) then $\bar{u} = -\frac{1}{\beta} p|_\Sigma$, where p is the solution to the equation*

$$-\frac{\partial p}{\partial t} - \Delta p = \bar{z} - z_d \quad \text{in } Q, \quad \frac{\partial p}{\partial n} = 0 \quad \text{on } \Sigma, \quad p(x,T) = \bar{z}(T) - z_d(T) \quad \text{in } \Omega. \tag{5.3.18}$$

Conversely, if a pair $(\tilde{z}, \tilde{p}) \in W(0,T; H^1(\Omega), (H^1(\Omega))') \times W(0,T; H^1(\Omega), (H^1(\Omega))')$ obeys the system

$$\begin{aligned} &\frac{\partial z}{\partial t} - \Delta z = f \quad \text{in } Q, \quad \frac{\partial z}{\partial n} = -\frac{1}{\beta} p|_\Sigma \quad \text{on } \Sigma, \quad z(x,0) = z_0 \quad \text{in } \Omega, \\ -&\frac{\partial p}{\partial t} - \Delta p = z - z_d \quad \text{in } Q, \quad \frac{\partial p}{\partial n} = 0 \quad \text{on } \Sigma, \quad p(T) = z(T) - z_d(T) \quad \text{in } \Omega, \end{aligned} \tag{5.3.19}$$

then the pair $(\tilde{z}, -\frac{1}{\beta}\tilde{p}|_\Sigma)$ is the optimal solution to problem (P_2).

Proof. We set $F_2(u) = J_2(z(f,u,z_0),u)$. A calculation similar to that of the previous section leads to:

$$F_2'(\bar{u})u = \int_Q (\bar{z} - z_d) w_u + \int_\Omega (\bar{z}(T) - z_d(T)) w_u(T) + \int_\Sigma \beta u \bar{u},$$

where w_u is the solution to the equation

$$\frac{\partial w}{\partial t} - \Delta w = 0 \quad \text{in } Q, \quad \frac{\partial w}{\partial n} = u \quad \text{on } \Sigma, \quad w(x,0) = 0 \quad \text{in } \Omega.$$

With formula (5.3.12) applied to p and w_u we obtain

$$\int_Q (\bar{z} - z_d) w_u + \int_\Omega (\bar{z}(T) - z_d(T)) w_u(T) = \int_\Sigma up.$$

Thus $F_2'(\bar{u}) = p|_\Sigma + \beta\bar{u}$. The end of the proof is similar to that of Theorem 5.2.4. ∎

5.4 Dirichlet boundary control

Now we want to control the heat equation by a Dirichlet boundary control, that is

$$\frac{\partial z}{\partial t} - \Delta z = f \quad \text{in } Q, \quad z = u \quad \text{on } \Sigma, \quad z(x,0) = z_0 \quad \text{in } \Omega. \tag{5.4.20}$$

Since we want to study equation (5.4.20) in the case when u belongs to $L^2(\Sigma)$, we have to define the solution to equation (5.4.20) by the transposition method. We follow the method introduced in Chapter 2. We first study the equation

$$\frac{\partial z}{\partial t} - \Delta z = 0 \quad \text{in } Q, \quad z = u \quad \text{on } \Sigma, \quad z(x,0) = 0 \quad \text{in } \Omega. \tag{5.4.21}$$

Suppose that u is regular enough to define the solution to equation (5.4.21) in a classical sense. Let y be the solution to

$$-\frac{\partial y}{\partial t} - \Delta y = \phi \quad \text{in } Q, \quad y = 0 \quad \text{on } \Sigma, \quad y(x,T) = 0 \quad \text{in } \Omega. \tag{5.4.22}$$

With a Green formula (which is justified if z and y are regular enough), we can write

$$\int_Q z\phi = -\int_\Sigma u\frac{\partial y}{\partial n} = \langle u, \Lambda\phi\rangle_{L^2(\Sigma)},$$

where $\Lambda\phi = -\frac{\partial y}{\partial n}$. Due to Theorem 5.2.2 we know that the mapping

$$\phi \longmapsto y$$

is linear and continuous from $L^2(Q)$ into $L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))\cap H^1(0,T;L^2(\Omega))$. Thus the operator Λ is linear and continuous from $L^2(Q)$ into $L^2(0,T;L^2(\Gamma))$, and Λ^* is a linear and continuous operator from $L^2(0,T;L^2(\Gamma))$ into $L^2(Q)$. Since the identity $\int_Q z\phi = \langle u, \Lambda\phi\rangle_{L^2(\Sigma)} = \langle \Lambda^* u, \phi\rangle_{L^2(Q)}$ is satisfied for every $\phi \in L^2(Q)$, we have $z = \Lambda^* u$. For $u \in L^2(\Sigma)$, the solution z_u to equation (5.4.21) is defined by $z_u = \Lambda^* u$. For equation (5.4.20) the definition of solution is stated below.

Definition 5.4.1 *A function $z \in L^2(Q)$ is a solution to equation (5.4.20) if, and only if,*

$$\int_Q z\phi = \int_Q fy + \int_\Omega z_0 y(0) - \int_\Sigma u\frac{\partial y}{\partial n}$$

for all $\phi \in L^2(Q)$, where y is the solution to equation (5.4.22).

Due to the continuity property of Λ^*, we have the following theorem.

Theorem 5.4.1 *For every $f \in L^2(Q)$, every $u \in L^2(\Sigma)$, and every $z_0 \in L^2(\Omega)$, equation (5.4.20) admits a unique weak solution $z(f,u,z_0)$ in $L^2(0,T;L^2(\Omega))$, moreover the operator*

$$(f,u,z_0) \mapsto z(f,u,z_0)$$

is linear and continuous from $L^2(Q)\times L^2(\Sigma)\times L^2(\Omega)$ into $L^2(Q)$.

5.4.1 Observation in $L^2(Q)$

Thanks to Theorem 5.4.1 we can study the following control problem

$$(P_3) \qquad \inf\{J_3(z,u) \mid (z,u)\in L^2(0,T;L^2(\Omega))\times L^2(\Sigma),\ (z,u) \text{ satisfies } (5.4.20)\},$$

with

$$J_3(z,u) = \frac{1}{2}\int_Q (z-z_d)^2 + \frac{\beta}{2}\int_\Sigma u^2.$$

We here suppose that z_d belongs to $L^2(Q)$. Contrary to the case of Neumann boundary controls, we cannot include an observation of $z(T)$ in $L^2(\Omega)$ in the definition of (P_3). To write optimality conditions for (P_3), we consider adjoint equations of the form

$$-\frac{\partial p}{\partial t} - \Delta p = g \quad \text{in } Q, \quad p = 0 \quad \text{on } \Sigma, \quad p(x,T) = 0 \quad \text{in } \Omega. \tag{5.4.23}$$

Theorem 5.4.2 *If $u \in L^2(\Sigma)$, then the solution z of equation (5.4.21) and the solution p of (5.4.23) satisfy the following formula*

$$\int_Q f\, p = \int_Q z\, g + \int_\Sigma u \frac{\partial p}{\partial n}. \tag{5.4.24}$$

Proof. The result directly follows from definition 5.4.1. ∎

Theorem 5.4.3 *Assume that $f \in L^2(Q)$, $z_0 \in L^2(\Omega)$, and $z_d \in L^2(0,T;L^2(\Omega))$. Let $(\bar z, \bar u)$ be the unique solution to problem (P_3). The optimal control $\bar u$ is defined by $\bar u = \frac{1}{\beta}\frac{\partial p}{\partial n}$, where p is the solution to the equation*

$$-\frac{\partial p}{\partial t} - \Delta p = \bar z - z_d \quad in\ Q, \quad p = 0 \quad on\ \Sigma, \quad p(x,T) = 0 \quad in\ \Omega. \tag{5.4.25}$$

This necessary optimality condition is also sufficient.

Proof. We set $F_3(u) = J_3(z(f, z_0, u), u)$. Due to Theorem 5.4.2, we have

$$F_3'(\bar u)u = \int_Q (\bar z - z_d) w_u + \beta \int_\Sigma \bar u u = \int_\Sigma \big(- \frac{\partial p}{\partial n} + \beta \bar u\big) u.$$

The end of the proof is now classical. ∎

5.4.2 Observation in $C([0,T];H^{-1}(\Omega))$

Denote by $\|\cdot\|_{H^{-1}(\Omega)}$ the dual norm of the $H_0^1(\Omega)$-norm, that is the usual norm in $H^{-1}(\Omega)$:

$$\|f\|_{H^{-1}(\Omega)} = \sup_{z \in H_0^1(\Omega)} \frac{\langle f, z\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}}{\|z\|_{H_0^1(\Omega)}}.$$

Let f be in $H^{-1}(\Omega)$ and denote by $(-\Delta)^{-1} f$ the solution to the equation

$$-\Delta z = f \quad \text{in } \Omega, \quad z = 0 \quad \text{on } \Gamma.$$

Theorem 5.4.4 *The mapping*

$$f \longmapsto |||f|||_{H^{-1}(\Omega)} = \langle f, (-\Delta)^{-1} f\rangle^{1/2}_{H^{-1}(\Omega)\times H_0^1(\Omega)}$$

is a norm in $H^{-1}(\Omega)$ equivalent to the usual norm.

Proof. We know that $(-\Delta)^{-1}$ is an isomorphism from $H^{-1}(\Omega)$ to $H_0^1(\Omega)$. Thus $f \mapsto \|(-\Delta)^{-1} f\|_{H_0^1(\Omega)}$ is a norm in $H^{-1}(\Omega)$ equivalent to the usual norm. If $f \in H^{-1}(\Omega)$, multiplying the equation $-\Delta((-\Delta)^{-1} f) = f$ by $(-\Delta)^{-1} f$, with a Green formula, we have

$$\int_\Omega |\nabla((-\Delta)^{-1} f)|^2 = \langle f, (-\Delta)^{-1} f\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} \le \|(-\Delta)^{-1} f\|_{H_0^1(\Omega)} \|f\|_{H^{-1}(\Omega)}.$$

Since the norm $f \mapsto \|(-\Delta)^{-1} f\|_{H_0^1(\Omega)}$ is equivalent to the norm in $H^{-1}(\Omega)$, we obtain

$$c_1 \|f\|^2_{H^{-1}(\Omega)} \le \langle f, (-\Delta)^{-1} f\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} \le c_2 \|f\|^2_{H^{-1}(\Omega)}.$$

The proof is complete. ∎

Theorem 5.4.5 *(i) Let $z(f,u,z_0)$ be the solution to equation (5.4.20). The operator*

$$(f,u,z_0)\mapsto z(f,u,z_0),$$

is linear and continuous from $L^2(Q)\times L^2(\Sigma)\times L^2(\Omega)$ into $C([0,T];H^{-1}(\Omega))$.
(ii) If $u\in L^2(\Sigma)$, and if $p_T\in H^1_0(\Omega)$, then the solution z of equation (5.4.21) and the solution p of

$$-\frac{\partial p}{\partial t}-\Delta p=0 \quad in\ Q, \quad p=0 \quad on\ \Sigma, \quad p(x,T)=p_T \quad in\ \Omega,$$

satisfy the following formula

$$\langle z(T),p_T\rangle_{H^{-1}(\Omega)\times H^1_0(\Omega)}=-\int_\Sigma u\frac{\partial p}{\partial n}. \tag{5.4.26}$$

Proof. (i) We only need to prove the regularity result for the solution z of equation (5.4.21). For every $\varphi\in H^1_0(\Omega)$ and every $\tau\in]0,T]$, consider the solution y to equation

$$-\frac{\partial y}{\partial t}-\Delta y=0 \quad \text{in } Q, \quad y=0 \quad \text{on } \Sigma, \quad y(\tau)=\varphi \quad \text{in } \Omega.$$

Due to Theorem 5.2.2, we have

$$\|y\|_{L^2(0,\tau;H^2(\Omega)\cap H^1_0(\Omega))}\le c\|\varphi\|_{H^1_0(\Omega)},$$

and the constant c is independent of τ. Let $(u_n)_n\subset L^2(\Sigma)$ a sequence of regular functions satisfying the compatibility condition $u_n(x,0)=0$, and converging to u in $L^2(\Sigma)$. Denote by z_n the solution to (5.4.21) corresponding to u_n. Since z_n is regular, it satisfies the formula

$$\int_\Omega z_n(\tau)\varphi=-\int_{\Gamma\times(0,\tau)}u_n\frac{\partial y}{\partial n}.$$

Thus we have

$$\|z_n(\tau)\|_{H^{-1}(\Omega)}=\sup_{\|\varphi\|_{H^1_0(\Omega)}=1}\Big|\int_{\Gamma\times(0,\tau)}u_n\frac{\partial y}{\partial n}\Big|\le c\|u_n\|_{L^2(\Sigma)},$$

where the constant c is independent of τ. From this estimate it follows that

$$\|z_n-z_m\|_{C([0,T];H^{-1}(\Omega))}=\|z_n-z_m\|_{L^\infty(0,T;H^{-1}(\Omega))}\le c\|u_n-u_m\|_{L^2(\Sigma)}.$$

Therefore the sequence $(z_n)_n$ converges to some $\tilde z$ in $C([0,T];H^{-1}(\Omega))$. Due to Theorem 5.4.1, the sequence $(z_n)_n$ converges to the solution z of equation (5.4.21). We finally have $z=\tilde z\in C([0,T];H^{-1}(\Omega))$.

(ii) Formula (5.4.26) can be established for regular data, and next deduced in the general case from density arguments. ∎

Now we are in position to study the control problem

$$(P_4) \qquad \inf\{J_4(z,u)\mid (z,u)\in L^2(0,T;L^2(\Omega))\times L^2(\Sigma),\ (z,u)\text{ satisfies (5.4.20)}\},$$

with

$$J_4(z,u)=\frac{1}{2}\|z(T)-z_T\|^2_{H^{-1}(\Omega)}+\frac{\beta}{2}\int_\Sigma u^2.$$

The proof of existence and uniqueness of solution to problem (P_4) is standard (see exercise 5.5.3).

Theorem 5.4.6 *Assume that* $f \in L^2(Q)$, $z_0 \in L^2(\Omega)$, *and* $z_d \in L^2(0,T;L^2(\Omega))$. *Let* $(\bar z, \bar u)$ *be the unique solution to problem* (P_4). *The optimal control* u *is defined by* $u = \frac{1}{\beta}\frac{\partial p}{\partial n}$, *where* p *is the solution to the equation*

$$-\frac{\partial p}{\partial t} - \Delta p = 0 \quad in\ Q, \quad p = 0 \quad on\ \Sigma, \quad p(x,T) = (-\Delta)^{-1}(\bar z(T) - z_T) \quad in\ \Omega. \tag{5.4.27}$$

Proof. We set $F_4(u) = J_4(z(f, z_0, u), u)$. If w_u is the solution to equation 5.4.21, and p the solution to equation 5.4.27, with the formula stated in Theorem 5.4.5(ii), we have

$$F_4(\bar u)u = \langle w_u(T), (-\Delta)^{-1}(\bar z(T) - z_T)\rangle_{H^{-1}(\Omega)\times H^1_0(\Omega)} + \beta\int_\Sigma \bar u u.$$

$$= \int_\Sigma \Big(-\frac{\partial p}{\partial n} + \beta\bar u\Big)u.$$

The proof is complete. ■

5.5 Exercises

Exercise 5.5.1

The notation are the ones of section 5.2. Let $(u_n)_n$ be a sequence in $L^2(0,T;L^2(\omega))$, converging to u for the weak topology of $L^2(0,T;L^2(\omega))$. Let z_n be the solution to equation (5.2.1) corresponding to u_n, and z_u be the solution to equation (5.2.1) corresponding to u. Prove that $(z_n(T))_n$ converges to $z_u(T)$ for the weak topology of $L^2(\Omega)$. Prove that the control problem (P_1) admits a unique solution.

Exercise 5.5.2

Prove that the control problem (P_2) of section 5.3 admits a unique solution.

Exercise 5.5.3

The notation are the ones of section 5.4. Let $(u_n)_n$ be a sequence in $L^2(\Sigma)$, converging to u for the weak topology of $L^2(\Sigma)$. Let z_n be the solution to equation (5.4.20) corresponding to u_n, and z_u be the solution to equation (5.4.20) corresponding to u. Prove that $(z_n(T))_n$ converges to $z_u(T)$ for the weak topology of $H^{-1}(\Omega)$. Prove that the control problem (P_4) admits a unique solution.

Exercise 5.5.4

Let Ω be a bounded domain in $\mathbb{R}^N$, with a boundary Γ of class C^2. Let $T > 0$, set $Q = \Omega\times(0,T)$ and $\Sigma = \Gamma \times (0,T)$. We consider a convection-diffusion equation with a distributed control

$$\frac{\partial z}{\partial t} - \Delta z + \vec V \cdot \nabla z = f + \chi_\omega u \quad \text{in } Q, \quad z = 0 \quad \text{on } \Sigma, \quad z(x,0) = z_0 \quad \text{in } \Omega. \tag{5.5.28}$$

The function f belongs to $L^2(Q)$, χ_ω is the characteristic function of ω, ω is an open subset of Ω, and the function u is a control variable. We suppose that $\vec{V} \in (L^\infty(Q))^N$. We want to study the control problem

$$(P_5) \qquad \inf\{J_5(z,u) \mid (z,u) \in C([0,T];L^2(\Omega)) \times L^2(0,T;L^2(\omega)),\ (z,u) \text{ satisfies } (5.5.28)\},$$

where

$$J_5(z,u) = \frac{1}{2}\int_Q (z-z_d)^2 + \frac{1}{2}\int_\Omega (z(T)-z_d(T))^2 + \frac{\beta}{2}\int_Q \chi_\omega u^2,$$

and $\beta > 0$. We assume that $z_d \in C([0,T];L^2(\Omega))$.

We first study equation (5.5.28) by a fixed point method. For that we need a regularity for the heat equation that we state below.

Regularity result. *For any $1 < q < \infty$, there exists a constant $C(q)$ such that the solution z to the heat equation*

$$\frac{\partial z}{\partial t} - \Delta z = f \quad \text{in } Q, \quad z = 0 \quad \text{on } \Sigma, \quad z(x,0) = 0 \quad \text{in } \Omega,$$

satisfies

$$\|z\|_{C([0,T];L^2(\Omega))} + \|z\|_{L^2(0,T;H_0^1(\Omega))} \leq C(q)\|f\|_{L^q(0,T;L^2(\Omega))} \quad \text{for all} \quad f \in L^q(0,T;L^2(\Omega)).$$

1 - Now we choose $1 < q < 2$. Let r be defined by $\frac{1}{2} + \frac{1}{r} = \frac{1}{q}$, and $\bar{t} \in]0,T]$ such that $C(q)\bar{t}^{1/r}\|\vec{V}\|_{(L^\infty(Q))^N} \leq \frac{1}{2}$. Let $\phi \in C([0,\bar{t}];L^2(\Omega)) \cap L^2(0,\bar{t};H_0^1(\Omega))$, and denote by z_ϕ the solution to equation

$$\frac{\partial z}{\partial t} - \Delta z = f + \chi_\omega u - \vec{V}\cdot\nabla\phi \quad \text{in } Q, \quad z = 0 \quad \text{on } \Sigma, \quad z(x,0) = z_0 \quad \text{in } \Omega. \tag{5.5.29}$$

Prove that the mapping

$$\phi \longmapsto z_\phi$$

is a contraction in $C([0,\bar{t}];L^2(\Omega)) \cap L^2(0,\bar{t};H_0^1(\Omega))$.

2 - Let $\hat{z}$ be the solution in $C([0,\bar{t}];L^2(\Omega)) \cap L^2(0,\bar{t};H_0^1(\Omega))$ to equation

$$\frac{\partial z}{\partial t} - \Delta z + \vec{V}\cdot\nabla z = f + \chi_\omega u \quad \text{in } \Omega \times (0,\bar{t}), \quad z = 0 \quad \text{on } \Gamma \times (0,\bar{t}), \quad z(x,0) = z_0 \quad \text{in } \Omega.$$

The existence of $\hat{z}$ follows from the previous question. Let $\phi \in C([0,2\bar{t}];L^2(\Omega)) \cap L^2(0,2\bar{t};H_0^1(\Omega))$ such that $\phi = \hat{z}$ on $[0,\bar{t}]$, and denote by z_ϕ the solution to equation

$$\frac{\partial z}{\partial t} - \Delta z = f + \chi_\omega u - \vec{V}\cdot\nabla\phi \quad \text{in } Q, \quad z = 0 \quad \text{on } \Sigma, \quad z(x,0) = z_0 \quad \text{in } \Omega. \tag{5.5.30}$$

Prove that the mapping

$$\phi \longmapsto z_\phi$$

is a contraction in the metric space

$$\{\phi \in C([0,2\bar{t}];L^2(\Omega)) \cap L^2(0,2\bar{t};H_0^1(\Omega)) \mid \phi = \hat{z} \text{ on } [0,\bar{t}]\},$$

for the metric corresponding to the norm of the space $C([0,2\bar{t}];L^2(\Omega)) \cap L^2(0,2\bar{t};H_0^1(\Omega))$.

3 - Prove that equation (5.5.28) admits a unique solution in $C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$, and that this solution obeys

$$\|z\|_{C([0,T];L^2(\Omega))} + \|z\|_{L^2(0,T;H_0^1(\Omega))} \le C(\|f\|_{L^2(Q)} + \|u\|_{L^2(\omega\times(0,T))} + \|z_0\|_{L^2(\Omega)}).$$

4 - Prove that the control problem (P_5) admits a unique solution. Write first order optimality conditions.

Exercise 5.5.5

Let Ω be a bounded domain in $\mathbb{R}^N$, with a boundary Γ of class C^2. Let $T>0$, set $Q=\Omega\times(0,T)$ and $\Sigma=\Gamma\times(0,T)$. We consider the heat equation with a control in a coefficient

$$\begin{cases} \frac{\partial y}{\partial t} - \Delta y + u\,y = f \text{ in } Q,\ T>0, \\ y=0 \text{ on } \Gamma\times]0,T[, \quad y(x,0)=y_0(x) \text{ in } \Omega, \end{cases} \tag{5.5.31}$$

avec $f\in L^2(Q)$, $y_0\in L^2(\Omega)$ et

$$u\in U_{ad}=\{u\in L^\infty(Q) \mid 0\le u(x,t)\le M \text{ a.e. in } Q\}, \quad M>0.$$

We want to study the control problem

$$(P_6) \qquad \inf\{J_6(y) \mid u\in U_{ad},\ (y,u) \text{ satisfies } (5.5.31)\}$$

avec $J_6(y)=\int_\Omega |y(x,T)-y_d(x)|^2dx$, y_d is a given function in $L^2(\Omega)$.

1 - Prove that equation (5.5.31) admits a unique solution y_u in $C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ (the fixed point method of the previous exercise can be adapted to deal with equation (5.5.31)). Prove that this solution belongs to $W(0,T;H_0^1(\Omega),H^{-1}(\Omega))$.

2 - Let $(u_n)_n\subset U_{ad}$ be a sequence converging to u for the weak star topology of $L^\infty(Q)$. Prove that $(y_{u_n})_n$ converges to y_u for the weak topology of $W(0,T;H_0^1(\Omega),H^{-1}(\Omega))$. Prove that (P_6) admits solutions.

3 - Let u and v be two functions in U_{ad}. Set $z_\lambda=(y_{u+\lambda v}-y_u)/\lambda$. Prove that $(z_\lambda)_\lambda$ converges, when λ tends to zero, to the solution $z_{u,v}$ of the equation

$$\begin{cases} \frac{\partial z}{\partial t} - \Delta z + v y_u + u z = 0 \text{ in } \Omega\times]0,T[, \\ z=0 \text{ on } \Gamma\times]0,T[, \quad z(x,0)=0 \text{ in } \Omega. \end{cases} \tag{5.5.32}$$

4 - Let (y_u,u) be a solution to problem (P_6). Write optimality conditions for (y_u,u) in function of $z_{u,v-u}$ (for $v\in U_{ad}$). Next, write this optimality condition by introducing the adjoint state associated with (y_u,u).