Hyperbolic Conservation laws

Theory and Numerics

by

G.D. Veerappa Gowda

TIFR Centre for Applicable Mathematics, Bangalore

1 Conservation laws

Many practical problems in science and engineering involve conservative quantities and lead to partial differential equations of this class. Scalar conservation law in one space dimension is of the form

$$u_t + f(u)_x = 0, \ \ (x,t) \in I\!R \times (0,\infty). \tag{1.1}$$

It is a simplest model of a nonlinear wave equation. Here $f : I\!R \to I\!R$ is a smooth function. Equation (1.1) must be augumented by some initial conditions and also possibly boundary conditions on a bounded spatial domain. The simplest problem is the pure initial value problem in which (1.1) holds with initial condition,

$$u(x,0) = u_0(x), \ \ x \in I\!R. \tag{1.2}$$

Let u be a smooth solution of (1.1) so that we can write (1.1) in a non conservative form

$$u_t + f'(u)u_x = 0. \tag{1.3}$$

The characteristic curve associated with (1.3) are defined as the solution of the differential equation

$$\left\{ \begin{array}{rcl} \frac{dx(t)}{dt} & = & f'(u(x(t),t)), \\ \\ x(0) & = & x_0. \end{array} \right. \tag{1.4}$$

Solution for (1.4) exists at least on a small interval $[0, t_0)$. Along such a curve, u is constant since

$$\begin{aligned} \frac{d}{dt}u(x(t),t) &= \frac{\partial u}{\partial t}(x(t),t) + \frac{\partial u}{\partial x}(x(t),t)\frac{dx(t)}{dt} \\ &= \left(\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial x}\right)(x(t),t) = 0. \end{aligned}$$

Now from (1.4) it follows that characteristic curves are straight lines where slopes are constants depending on the initial data. These curves are given by

$$x(t) = x_0 + tf'(u(x_0)) \tag{1.5}$$

This important property gives a way to construct smooth solutions. Hence

$$u(x(t),t) = u_0(x_0). \tag{1.6}$$

Therefore if u is smooth, then u can be find by the **method of characteristics**.

Example 1. Consider the linear equation $u_t + au_x = 0$, a is a real number. By the method of characteristics, solution is given by

$$u(x,t) = u_0(x - at).$$

1.1 Existence of a non-smooth solution:

Suppose f is nonlinear, say $f'' > 0$. Assume that there exists two points $x_1 < x_2$ such that

$$m_1 = \frac{1}{f'(u_0(x_1))} < m_2 = \frac{1}{f'(u_0(x_2))}.$$

Then the characteristics $C_1 = x_1(t)$ and $C_2 = x_2(t)$ drawn from the points $(x_1, 0)$ and $(x_2, 0)$ respectively, have slopes m_1 and m_2 and intersect necessarily at some point P. At this point P, the solution u will take both the values $u_0(x_1)$ and $u_0(x_2)$, which is clearly impossible if u is continuous. Hence, solution u can not be continuous at the point P. Note that this phenomenon is independent of the smoothness of the functions u_0 and f

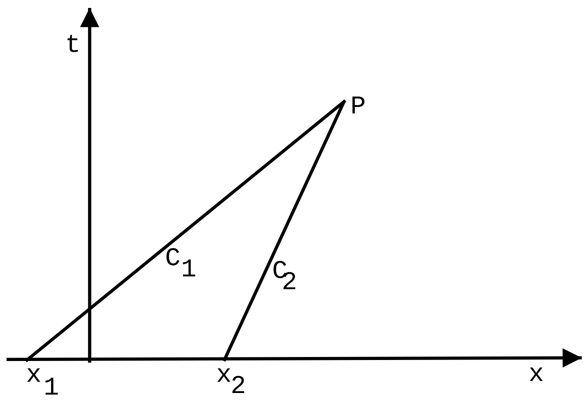

Fig. 1.1

Example 2. Consider the Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

with $u(x,0) = u_0(x) = \begin{cases} 1 & \text{if} \quad x < 0 \\ 1 - x & \text{if} \quad 0 \le x \le 1 \\ 0 & \text{if} \quad x > 1 \end{cases}$

By using the method of characteristics, we can solve upto the time when the characteristics intersects. By (1.5) we have

$$x = x(x_0, t) = x_0 + tu_0(x_0)$$

so that

$$x = x(x_0, t) = \begin{cases} x_0 + t & \text{if} \quad x_0 \leq 0 \\ x_0 + t(1 - x_0) & \text{if} \quad 0 \leq x_0 \leq 1 \\ x_0 & \text{if} \quad x_0 \geq 1 \end{cases}$$

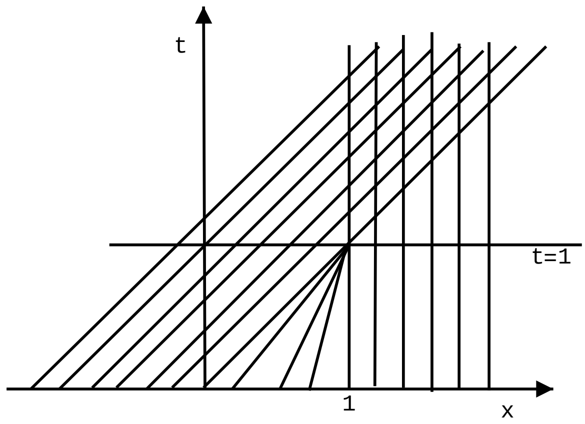

Fig. 1.2

For $t < 1$, the characteristics do not intersect. Hence given a point (x, t) with $t < 1$, we draw the backward characteristic passing through this point, and we detemine the corresponding x_0.

$$x_0 = \begin{cases} x - t & \text{if} \quad x \leq t \\ \frac{x-t}{1-t} & \text{if} \quad t \leq x \leq 1 \\ x & \text{if} \quad x \geq 1 \end{cases}$$

$$\Rightarrow u(x, t) = \begin{cases} 1 & \text{if} \quad x \leq t \leq 1 \\ \frac{1-x}{1-t} & \text{if} \quad t \leq x \leq 1 \\ 0 & \text{if} \quad x \geq 1, t < 1 \end{cases}$$

At $t = 1$, the characteristics intersect. So we can say at $t = 1$,

$$u(x, 1) = \begin{cases} 1 & \text{if} \quad x < 1 \\ 0 & \text{if} \quad x > 1. \end{cases}$$

Now it is clear that discontinuities may develop after a finite time if f is nonlinear, even when u_0 is smooth. These consideration lead us to introduce **weak solutions**.

1.2 Weak Solutions; Rankine-Hugoniot Condition:

Let $v : I\!R \times [0, \infty) \to I\!R$ is a smooth function with compact support. Multiply (1.1) by v and integration by parts:

$$\begin{aligned} 0 &= \int_0^\infty \int_{-\infty}^\infty (u_t + f(u)_x) v \, dx \, dt \\ &= -\int_0^\infty \int_{-\infty}^\infty u v_t dx dt - \int_0^\infty \int_{-\infty}^\infty f(u) v_x dx dt - \int_{-\infty}^\infty uv|_{t=0} dx. \end{aligned} \tag{1.7}$$

Now note that RHS of (1.7) is valid for any bounded measurable function u.

<u>Definition:</u> We say that $u \in L^\infty(I\!R \times [0, \infty))$ is a weak solution of (1.1) and (1.2) provided

$$\int_0^\infty \int_{-\infty}^\infty (uv_t + f(u)v_x) dx dt + \int_{-\infty}^\infty uv|_{t=0} = 0 \tag{1.8}$$

for all smooth function v with compact support (usually v is called test function).

Suppose u satisfies (1.8) and u is smooth, then u satisfies (1.1). Solution of (1.8) although not continuous, has a simple structure that we can see now.

Let $x(t)$ be a curve in the (x, t) plane across which $u(x, t)$ fails to be continuous, but such that u has a limit as points (x, t) approaches the curve from either side. Suppose in some openset $V \subset I\!R \times (0, \infty)$, u is smooth on either side of the smooth curve $x(t)$. Let V_l be the part of V on the left of the curve and V_r that part on the right. Assume u and its first derivatives are uniformly continuous on V_l and V_r.

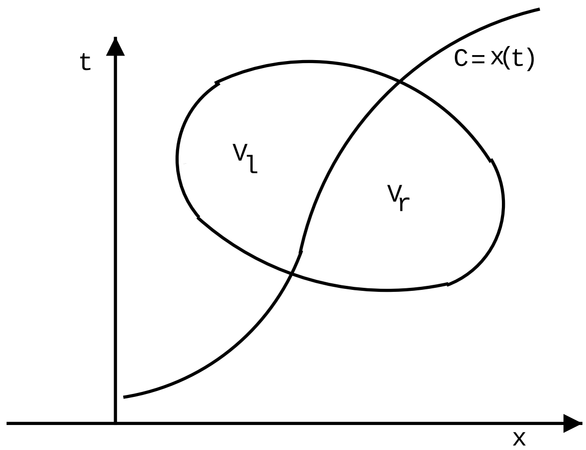

Fig. 1.3

First choose a test function v with compact support in V_l. Then (1.8) become

$$0 = \int_0^\infty \int_{-\infty}^\infty (uv_t + f(u)v_x)dxdt = -\int_0^\infty \int_{-\infty}^\infty (u_t + f(u)_x)v\, dxdt, \tag{1.9}$$

the integration by parts being justified since u is C^1 in V_l and v vanishes near the boundary of V_l. The identity (1.9) holds for all test functions v with compact support in V_l and so

$$u_t + f(u)_x = 0 \text{ in } V_l. \tag{1.10}$$

Similarly

$$u_t + f(u)_x = 0 \text{ in } V_r \tag{1.11}$$

Now choose a test function v with compact support in V but which does not necessarily vanish along the curve $x(t)$.

Again using (1.8) we have

$$\begin{aligned} 0 &= \int_0^\infty \int_{-\infty}^\infty (uv_t + f(u)v_x)dxdt \\ &= \iint_{V_l} (uv_t + f(u)v_x)dxdt \\ &\quad + \iint_{V_r} (uv_t + f(u)v_x)dxdt \end{aligned}$$

Now since v has a compact support with in V, we have by (1.10)

$$\begin{aligned} \iint_{V_l} (uv_t + f(u)v_x)dxdt &= -\iint_{V_l} (u_t + f(u)_x)v\, dxdt \\ &\quad + \int_C (u_l\nu_2 + f(u_l)\nu_1)v\, ds \\ &= \int_C (u_l\nu_2 + f(u_l)\nu_1)v\, ds \end{aligned} \tag{1.12}$$

Here $\nu = (\nu_1, \nu_2)$ is the unit normal to the curve $C = x(t)$ pointing from V_l into V_r and $u_l = \lim_{\substack{(x,t)\to c \\ (x,t)\in V_l}} u(x,t)$. Similalry

$$\iint_{V_r} (uv_t + f(u)v_x)dxdt = -\int_C (u_r\nu_2 + f(u_r)\nu_1)v\, ds. \tag{1.13}$$

Here $u_r = \lim\limits_{\substack{(x,t)\to c\\(x,t)\in V_r}} u(x,t)$. By adding (1.12) and (1.13) we have

$$\int_C ((f(u_l) - f(u_r))\nu_1 + (u_l - u_r)\nu_2)\, v\, ds = 0 \text{ along } C.$$

We can take $(\nu_1, \nu_2) = \frac{1}{(1+\dot{x}(t)^2)^{1/2}}(1, -\dot{x}(t))$. Hence

$$(f(u_l) - f(u_r)) = \dot{x}(t)(u_l - u_r) \text{ along the curve } C = x(t). \tag{1.14}$$

Notation: $[u] = u_l - u_r =$ jump in u across the curve C
$[f(u)] = f(u_l) - f(u_r) =$ jump in $f(u)$
$s = \dot{x}(t) =$ speed of the discontinuity .

Employing this notation we write (1.14) as

$$[f(u)] = s[u] \tag{1.15}$$

along the discontinuity curve. Relation (1.15) is called the **Rankine-Hugnoit (R-H) condition**

Remark: Observe that the speed $s = \dot{x}(t)$ and the values $u_l, u_r f(u_l), f(u_r)$ will vary along the curve C. The point is that even though these quantities change, the expression $[f(u)] = s[u]$ must always exactly balance.

Example 2': Consider the same exaples 2. Observe that for $t \geq 1$ the method of characterstics breaks down, since the characteristics intersects. Now we should define u for $t > 1$ so that it satisfies the condition (1.15). Now solution

$$u(x,t) = \begin{cases} 1 & \text{if} \quad x < \frac{1+t}{2} \\ 0 & \text{if} \quad x > \frac{1+t}{2}. \end{cases} \tag{1.16}$$

$\Rightarrow$

$$u(x,1) = \begin{cases} 1 & \text{if} \quad x < 1 \\ 0 & \text{if} \quad x > 1 \end{cases}$$

and

$$\dot{x}(t) = s = \frac{1}{2}, \qquad f(u_l) - f(u_r) = \frac{1}{2} - 0 = \frac{1}{2}$$
$$s(u_l - u_r) = \frac{1}{2}$$

$\Rightarrow$ solution (1.16) satisfies R-H condition along the curve $C = x(t) = \frac{1+t}{2}$.

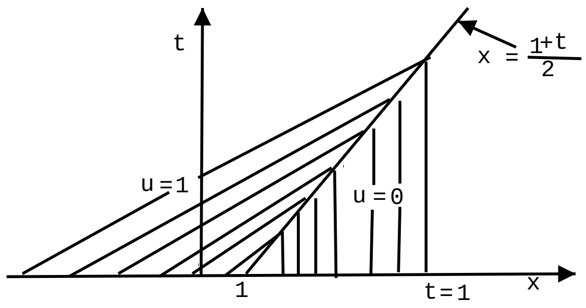

Fig. 1.4

Hence solution given by (1.16) is a weak solution for $t \geq 1$.

Non-uniqueness of weak solutions: Let us now try the similar problem with same techique.

Example 3 : Now consider the same Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

with $u(x,0) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x > 0. \end{cases}$

The method of characteristics this time does not lead to any ambiguity in defining u, but does fail to give any information on the Wedge $\{0 < x < t\}$(see Fig.1.5).

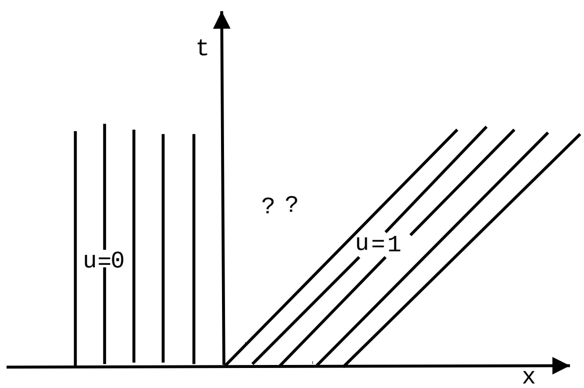

Fig. 1. 5

Now how to fill up this gap? We can fill this gap in the fashion of previous example(see Fig.1.6). Set

$$u_1(x,t) = \begin{cases} 0 & \text{for} \quad x < t/2 \\ 1 & \text{for} \quad t/2 < x \end{cases}$$

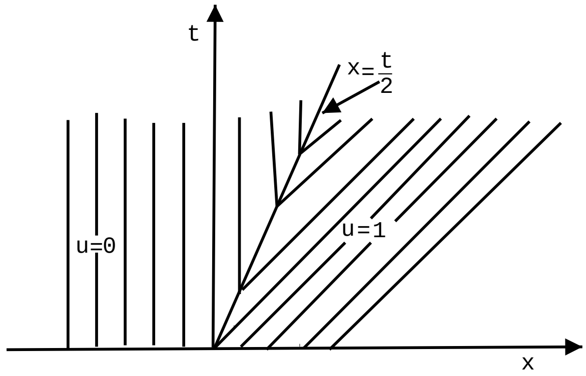

Fig. 1.6

Along the curve $C = x(t) = t/2$,

$$f(u_l) - f(u_r) = 0 - 1/2, \ s = \dot{x}(t) = 1/2, \ u_l - u_r = 0 - 1 = -1.$$

Therefore u_1 satisfies the R-H condition along C.

On the other hand we can fill this gap $\{0 < x < t\}$ by(see Fig.1.7)

$$u_2(x,t) = \begin{cases} 0 & \text{if} \quad x < 0 \\ \frac{x}{t} & \text{if} \quad 0 < x \leq t \\ 1 & \text{if} \quad x \geq t \end{cases}$$

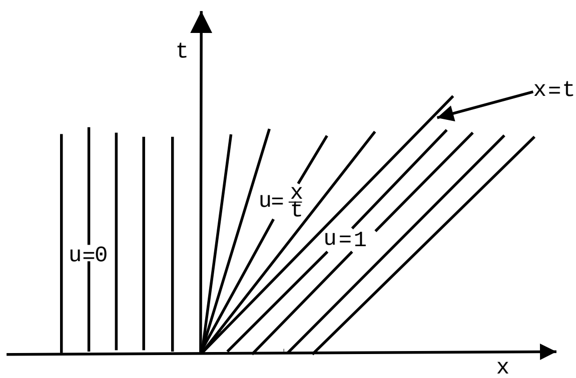

Fig. 1.7

Solution u_2 is continuous and hence u_2 is also a weak solution of the Burgers equation. Hence we need to find a criteria that enables us to choose the **"physically relevant solution"** among all the weak solutions of (1.1) and (1.2). **Physically relevant solution:** Let us look at the related viscous problem and will then let the viscosity tends to zero. The viscous equation is

$$\begin{cases} u_t + f(u)_x = \epsilon u_{xx} \\ u(x,0) = u_0(x) \end{cases} \tag{1.17}$$

Let $u^\epsilon(x,t)$ be the solution of (1.17) and $u = \lim_{\epsilon \to 0} u(x,t)$ in L^1-norm. Then u is a weak solution of (1.1) and (1.2). This u is called physically relevant solution of (1.1) and (1.2).

But how to pick up this physcially relevant solution. We want to find a simpler condition. Fig.1.4 suggests that 'A shock should have characteristics going into the shock, as time advances'.

A propagating discontinuity with characteristics coming out of it, as in Fig. 1.6, is unstable to perturbations, i.e., if we add some viscosity to the system, will cause this to be replaced by a rarefaction of characteristics as in Fig. 1.7. On this ground we like to reject the solution u_1 in Example 3. This gives us first version of Entropy Condition for convex flux function f:

Entropy Condition (for convex flux function): A dicsontinuity propagating with a speed s defined by (1.15) satisfies the entropy condition if

$$f'(u_l) > s > f'(u_r). \tag{1.18}$$

For a convex flux f, condition (1.18) is equivalent to

$$u_l > u_r \tag{1.19}$$

Now in Example 3, Solution $u_1(x,t)$ not satisfying condition 1.18, hence we reject this solution and keeping the continuous solution u_2.

A more general form of this condition, due to Oleinik, applies also to non convex flux functions f:

Entropy condition (Oleinik) (for all flux functions): A discontinuity propogating with a speed s defined by (1.15) satisfies the entropy condition if

$$\frac{f(v) - f(u_l)}{v - u_l} \geq s \geq \frac{f(v) - f(u_r)}{v - u_r}. \tag{1.20}$$

b

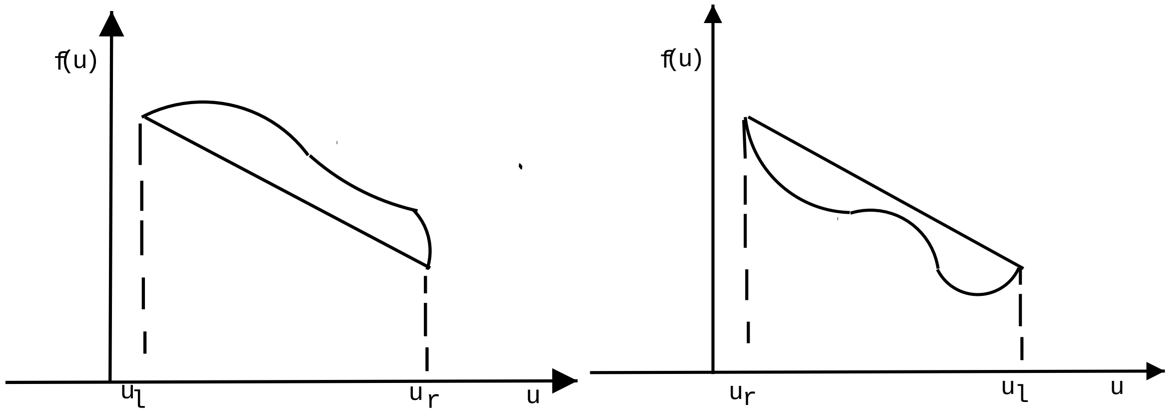

Fig. 1.8

for all v between u_l and u_r.

If f is convex (1.20) reduces to (1.18).

The condition (1.20) has the simple geometric interpretation: $f(u)$ must lie entirely to one side of the chord joining $(u_l, (fu_l))$ and $(u_r, f(u_r))$(see Fig.1.8). i.e., If $u_l < u_r$, then the graph of f over $[u_l, u_r]$ lies above the chord :

$$f(\alpha u_r + (1-\alpha)u_l) \geq \alpha f(u_r) + (1-\alpha)f(u_l) \text{ for } 0 \leq \alpha \leq 1 \tag{1.21}$$

If $u_r < u_l$, then the graph of f over $[u_r, u_l]$ lies below the chord :

$$f(\alpha u_r + (1-\alpha)u_l) \leq \alpha f(u_r) + (1-\alpha)f(u_l) \text{ for } 0 \leq \alpha \leq 1 \tag{1.22}$$

Also, Oleinik entropy condition is equivalent to the following Kruzkov entropy condition(see section 5):

Kruzkov entropy condition: A weak solution u of (1.1) and (1.2) is the entropy solution if

$$\int_0^\infty \int_{-\infty}^\infty \left[|u-l| \frac{\partial \phi}{\partial t} + sgn(u-l)(f(u)-f(l)) \frac{\partial \phi}{\partial x} \right] dxdt \geq 0 \tag{1.23}$$

for all real l and for all non negetive smooth functions ϕ with compact support.

2 Riemann Problem

A conservation law together with piecewise constant data having a single discontinuity is known as the Riemann Problem i.e.,

$$\text{Riemann Problem}: \begin{cases} u_t + f(u)_x = 0 \text{ in } I\!R \times (0, \infty) \\ u(x,0) = \begin{cases} u_l & \text{if} \quad x < 0 \\ u_r & \text{if} \quad x > 0 \end{cases} \end{cases} \tag{2.1}$$

Here $u_l, u_r \in I\!R$ are the left and right initial states.

Riemann problem for convex flux:

Theorem 2.1(Solution of a Riemann problem for convex f): Let f be strictly convex and $G = (f')^{-1}$.

(i) if $u_l > u_r$, then the weak solution satisfying entropy condition is given by

$$u(x,t) = \begin{cases} u_l & \text{if} \quad x < st \\ u_r & \text{if} \quad x > st \end{cases} \tag{2.2}$$

where $s = \frac{f(u_l) - f(u_r)}{u_l - u_r}$

(ii) if $u_l < u_r$, then the weak solution satisfying entropy condition is given by

$$u(x,t) = \begin{cases} u_l & \text{if} \quad x < tf'(u_l) \\ G\left(\frac{x}{t}\right) & \text{if} \quad tf'(u_l) < x < tf'(u_r) \\ u_r & \text{if} \quad x > tf'(u_r) \end{cases} \tag{2.3}$$

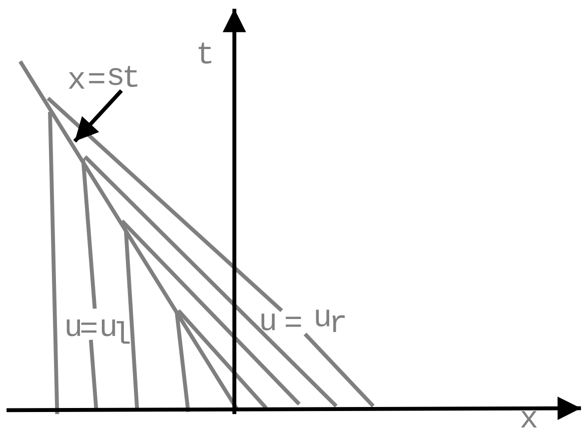

Fig. 2.1

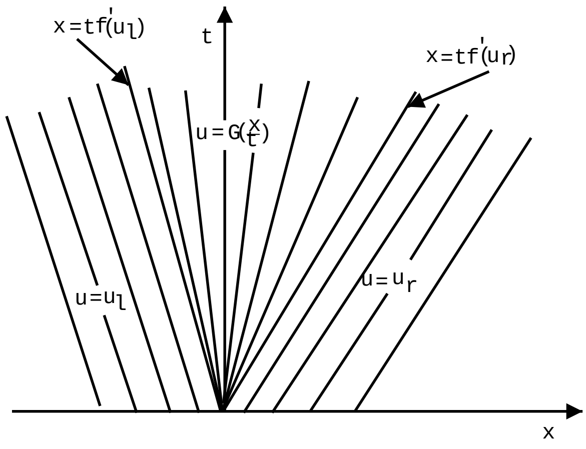

Fig. 2.2

Remark: In the first case the states u_l and u_r are separated by a shock wave with constant speed s. In the second case the state u_l and u_r are separated by a (centered) rarefaction wave.

Proof: (i) Across the shock $u(x,t)$ satisfies the Rankine-Hugoniot condition. If $x \neq st$ the constant states u_l and u_r are the solution of left and right part of the shock. Since $u_l > u_r$, the entropy condition holds as well.

(ii) it is enough to check that, solution defined in the region $\{f'(u_l) < x/t < f'(u_r)\}$ solves the equation in (2.1). Let $u(x,t) = v(x/t)$. then

$$\begin{aligned} 0 = u_t + f(u_x) &= u_t + f'(u)u_x \\ &= -v'\left(\frac{x}{t}\right)\frac{x}{t^2} + f'(v)v'\left(\frac{x}{t}\right)\cdot\frac{1}{t} \\ &= v'\left(\frac{x}{t}\right)\frac{1}{t}[f'(v) - \frac{x}{t}] \end{aligned}$$

$$\begin{aligned} &\Rightarrow \quad f'(v\left(\frac{x}{t}\right) = \frac{x}{t} \\ &\Rightarrow \quad u(x,t) = v\left(\frac{x}{t}\right) = (f')^{-1}\left(\frac{x}{t}\right) = G\left(\frac{x}{t}\right) \text{ solves (3.1).} \end{aligned}$$

Now $v\left(\frac{x}{t}\right) = u_l$ provided $\frac{x}{t} = f'(u_l)$ and similarly $v\left(\frac{x}{t}\right) = u_r$ if $\frac{x}{t} = f'(u_r)$.

$\Rightarrow$ Solutions defined in (2.3) is a continuous solution.

In the region $\{f'(u_l) < \frac{x}{t} < f'(u_r)\}$ solution satisfies

$$u(x+z,t) - u(x,t) \quad = \quad G(\frac{x+z}{t}) - G\left(\frac{x}{t}\right)$$

$$\leq \quad Lip(G)\left(\frac{z}{t}\right)$$

for every $z > 0$. This shows solution defined (2.3) is a weak solution satisfying entropy condition.

Riemann problem for general non canvex f: Let f be a non-convex function.

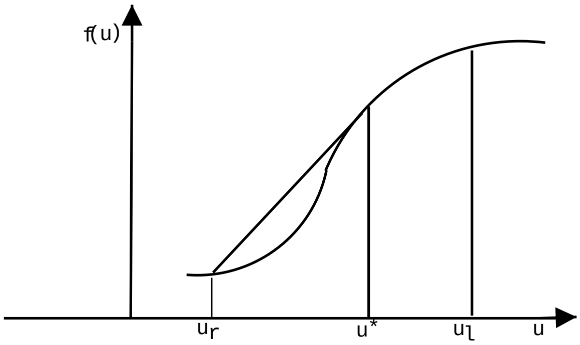

Fig. 2.3

Case (i): $u_l > u_r$ (See Fig.2.3)

Consider the convex hull of the set

$$A = \{(x, y)/u_r \leq x \leq u_l \ , \ y \leq f(x)\}.$$

(convex hull of the set A is the smallest convex set containing A). Look at the upper boundary of the set, it is composed of straight line segment $(u_r, f(u_r))$ to $(u^*, f(u^*))$ and then $y = f(x)$ upto $(u_l, f(u_l))$.

Connect u_r to u^* by a shock, u^* can be calculated by

$$f'(u^*) = \frac{f(u^*) - f(u_r)}{u^* - u_r}$$

and then connect u^* and u_l by a rarefaction wave(see Fig.2.4).

Note that across the line $x = tf'(u_l)$, solution is continuous and across the line $x = tf'(u^*)$, the solution is dicontinuous, with the left state u^* and the right state u_r

$$u(x,t) = \begin{cases} u_l & \text{if} \quad x \leq tf'(u_l) \\ (f')^{-1}\left(\frac{x}{t}\right) & \text{if} \quad tf'(u_l) \leq x \leq tf'(u^*) \\ u_r & \text{if} \quad x > tf'(u^*). \end{cases}$$

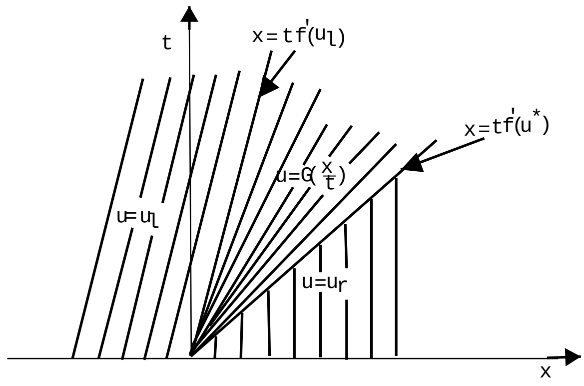

Fig. 2.4

(f is a concave function in Fig.2.3 between u^* and u_l, hence $G = (f')^{-1}$ exists).

Case (ii): $u_l < u_r$ (See Fig. 2.5)

In this case consider the convex hull of the set

$$A = \{(x, y)/u_l \leq x \leq u_r, y \geq f(x)\}$$

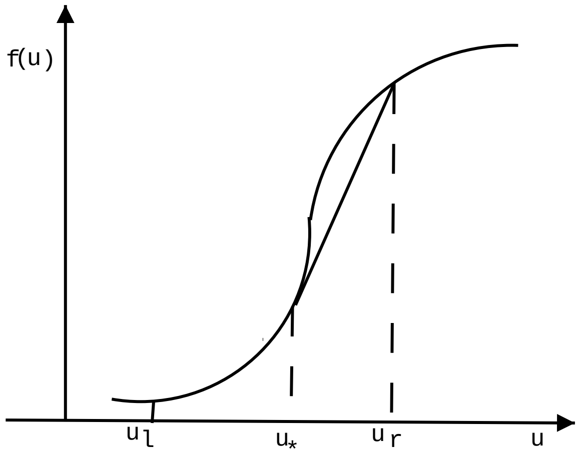

Fig. 2.5

Connect u_l and u_* by a rarefaction wave and then connect u_* and u_r by a shock where u_* is given by

$$f'(u_*) = \frac{f(u_*) - f(u_r)}{u_* - u_r}$$

.

$$u(x,t) = \begin{cases} u_l & \text{if} \quad x \le tf'(u_l) \\ (f')^{-1}\left(\frac{x}{t}\right) & \text{if} \quad tf'(u_l) \le x \le f'(u_*)t \\ u_r & \text{if} \quad x > tf'(u_r)t. \end{cases}$$

See Fig.2.6

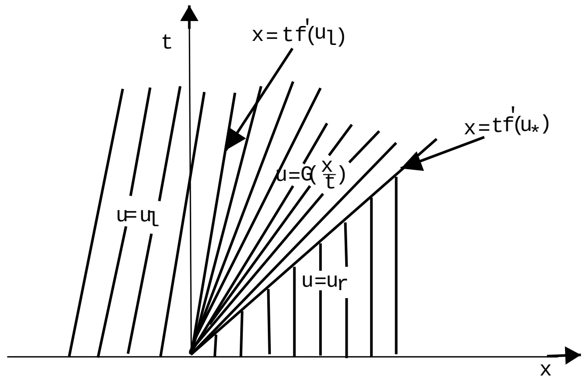

Fig. 2.6

3 Finite Differrence Schemes for Approximating Conservatin Laws

Here we restrict ourselves to explicit one step finite difference schemes. Descretize the $x-$axis by a sequence $\{x_{i+\frac{1}{2}}\}$ with $x_{i+\frac{1}{2}} = (i+\frac{1}{2})h, i \in Z, h > 0$ and $t-$axis by $\{t_n\}$ with $t_n = n\Delta t, n = 0, 1, 2, ..., \Delta t > 0$. Δt and h are called time step and spacial mesh size respectively. Let $\lambda = \frac{\Delta t}{h}$ and $x_i = \frac{x_{i+\frac{1}{2}} + x_{i-\frac{1}{2}}}{2}$.

In order to approximate the conservation law

$$\begin{cases} u_t + f(u)_x = 0 \text{ in } I\!R \times (0, \infty), \\ u(x,0) = u_0(x), \quad x \in I\!R \end{cases} \tag{3.1}$$

we introduce $(2k+1)$ point scheme of the form

$$v_i^{n+1} = H(v_{i-k}^n, v_{i-k+1}^n, ..., v_i^n, ..., v_{i+k-1}^n, v_{i+k}^n) \tag{3.2}$$

where $H : I\!R^{2k+1} \to I\!R$ is a continuous function and v_i^n denotes the approximation of the exact solution u at the grid point $(x_{i+\frac{1}{2}}, t_n)$. Initial data $\{v_i^0\}$ is defined by

$$v_i^0 = \frac{1}{h} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u_0(x)\, dx. \tag{3.3}$$

If $u_0(x)$ is continuous,then one can take

$$v_i^0 = u_i^0 = u_0(x_i) \ \forall i$$

Definition: A difference scheme (3.2) is said to be in the conservative form,if there exists a continuous function $F : I\!R^{2k} \to I\!R$ such that

$$H(v_{i-k}^n, ..., v_{i+k}^n) = v_i^n - \lambda(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n) \tag{3.4}$$

where $F_{i+\frac{1}{2}}^n = F(v_{i-k+1}^n, ..., v_{i+k}^n)$. The function F is called **numerical flux**.

Definition: The diffrence scheme (3.4) is said to be consistence with the equation (3.1) if

$$F(v, ..., v) = f(v) \ \ \forall v \in I\!R \tag{3.5}$$

$$i.e., \quad H(v, ..., v) = v \ \ \forall v \in I\!R \tag{3.6}$$

In order to analyze the convergence of the solution $\{v_i^n\}$ of the difference scheme (3.4) we introduce the piecewise constant function v_h defined a.e. in $I\!R \times (0, \infty)$ by

$$v_h(x,t) = v_i^n \ \text{ for } \ (x,t) \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times [t_n, t_{n+1}) \tag{3.7}$$

Theorem 4.1 (Lax-Wendroff)Let v_h be the numerical solution obtained from the scheme (3.2) which is in the conservative form and consisitent with equation (3.1). Assume that there exists a sequence $\{h_k\}$ which tends to 0 as $k \to \infty$ such that,if we set $\Delta_k t = \lambda h_k$ ($\lambda = \frac{\Delta_k t}{h_k}$, kept constant)
(i)$||v_{h_k}||_{L^\infty(I\!R\times(0,\infty))} \leq C$ for some constant $C > 0$,
(ii) v_{h_k} converges in $L^1_{loc}(I\!R \times (0, \infty))$ and a.e to a function v. Then v is a weak solution of (3.1).

Proof: Let $\phi \in C_0^1(I\!R \times I\!R_+)$ and let

$$\phi_i^n = \phi(x_i, t_n),$$

$$\phi_h(x,t) = \phi_i^n \quad \text{if} \quad (x,t) \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times [n\Delta t, (n+1)\Delta t).$$

Multiply equation (3.4) by ϕ_i^n and sum it over i and n,then we have

$$h\sum_{i,n}(v_i^{n+1} - v_i^n)\phi_i^n + \Delta t\sum_{i,n}(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n)\phi_i^n = 0.$$

A summation by parts gives

$$h\sum_i u_i^0\phi_i^0 + h\sum_{i,n} v_i^{n+1}(\phi_i^{n+1}-\phi_i^n) + \Delta t\sum_{i,n} F_{i+\frac{1}{2}}(\phi_{i+1}^n-\phi_i^n) = 0.$$

Now define a piecewise constant function F_h by

$$F_h(x,t) = F_{i+\frac{1}{2}}^n = F(v_{i-k+1}^n,...,v_{i+k}^n),(x,t)\in(x_i,x_{i+1})\times[t_n,t_{n+1})$$

Now since v_h is bounded in $L^\infty(I\!R\times(0,\infty))$ we have

$$\begin{aligned} h\sum_i v_i^0\phi_i^0 &= \int_{I\!R} v_h(x,0)\phi_h(x,0)dx \\ &= \int_{I\!R} v_h(x,0)\phi(x,0)dx + O(h) \end{aligned} \tag{3.8}$$

$$\begin{aligned} h\sum_{i,n} v_i^{n+1}(\phi_i^{n+1}-\phi_i^n) &= \int_{\Delta t}^{\infty}\int_{-\infty}^{\infty} v_h(x,t)\frac{\phi_h(x,t)-\phi_h(x,t-\Delta t)}{\Delta t}dxdt \\ &= \int_{\Delta t}^{\infty}\int_{-\infty}^{\infty} v_h(x,t)\frac{\partial\phi}{\partial t}dxdt + O(h), \end{aligned} \tag{3.9}$$

$$\begin{aligned} \Delta t\sum_{i,n} F_{i+\frac{1}{2}}(\phi_{i+1}^n-\phi_i^n) &= \int_0^{\infty}\int_{-\infty}^{\infty} F_h(x,t)\frac{\phi_h(x+h,t)-\phi_h(x,t)}{h}dxdt \\ &= \int_0^{\infty}\int_{-\infty}^{\infty} F_h(x,t)\frac{\partial\phi}{\partial x}dxdt + O(h). \end{aligned} \tag{3.10}$$

Observe first that

$$F_h(x,t) = F(v_h(x-(k-\frac{1}{2})h,t),...,v_h(x+(k-\frac{1}{2})h,t))$$

and set

$$w_h^i(x,t) = v_h(x+(i-\frac{1}{2})h,t), \quad -k\le i\le k.$$

Let K be a compact subset of $I\!R\times(0,\infty)$ and $K_1 = K+(i-\frac{1}{2})h$. Then

$$\begin{aligned} \int_K |w_h^i(x,t)-v(x,t)|\,dx\,dt &\le \int_{K_1} |v_h(x,t)-v(x,t)|\,dx\,dt \\ &\quad + \int_K |v(x+(i-\frac{1}{2})h,t)-v(x,t)|\,dx\,dt \end{aligned}$$

In the right side of the above inequality first term goes to 0 because v_h converges to v in L^1_{loc} and the second term goes to 0 by the property, continuity in mean of the function v. Therefore w_h^i converges to v in $L^1(K)$ for all $|i| \leq k$. Therfore we can exract a subsequence ,still denoted by w_h^i,which converges to v a.e in K. Since F is continuous we have,

$$F_h(x,t) = F(w_h^{-k+1}(x,t), ..., w_h^k(x,t)) \to F(v(x,t), ..., v(x,t))$$

a.e in K. Hence by consistency property (3.6), $F(v,..,v) = f(v)$. It follows from Lebesgue dominated convergence theorem we have

$$\int_0^{\infty}\int_{-\infty}^{\infty} F_h(x,t)\frac{\partial \phi}{\partial x}dxdt \ \to \int_0^{\infty}\int_{-\infty}^{\infty} f(v((x,t))\frac{\partial \phi}{\partial x}dxdt$$

. Since v_h converges to v in $L^1_{loc}(I\!R \times (0,\infty))$, then from (3.8) and (3.9),finally (3.8) gives(by going to further subsequence h_k, if necessary)

$$\int_{-\infty}^{\infty} u_0\phi(x,0)dx + \int_0^{\infty}\int_{-\infty}^{\infty} v\frac{\partial \phi}{\partial t}dxdt + \int_0^{\infty}\int_{-\infty}^{\infty} f(v)\frac{\partial \phi}{\partial x}dxdt = 0$$

$$\forall\ \phi \in C_0^1(I\!R \times I\!R_+).$$

$\Rightarrow v$ is a weak solution of (3.1).

Examples of 3-point scheme:

The entropy solution of the Riemann problem

$$\begin{cases} u_t + f(u)_x = 0 \ \text{in} \ I\!R \times (0,\infty) \\ u(x,0) = \begin{cases} u_l & \text{if} \quad x < 0 \\ u_r & \text{if} \quad x > 0 \end{cases} \end{cases} \tag{3.11}$$

is self-similiar,i.e., of the form

$$u(x,t) = w_R(\frac{x}{t}, u_l, u_r) \tag{3.12}$$

where w_R depends only on the function f and consists of two constant states u_l and u_r.

Example 3.1:The Godunov scheme:

Step-1: First we would like to solve exactly the following problem with piecewise constant initial data:

$$\begin{cases} w_t + f(w)_x = 0, \ \ t \in (t_n, t_{n+1}] \\ w(x,t_n) = v_h(x,t_n) \end{cases} \tag{3.13}$$

where $v_h(x,t_n) = v_i^n,\ x \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \forall i \in Z$. To solve this we decompose this problem into several local Riemann problem(see fig.below) centered at points $x_{i+\frac{1}{2}}$,for each $i \in Z$ i.e.,

$$\begin{cases} w_t + f(w)_x = 0,\ \ t \in (t_n, t_{n+1}] \\ w(x,t_n) = \begin{cases} v_i^n & \text{if} \quad x < 0 \\ v_{i+1}^n & \text{if} \quad x > 0 \end{cases} \end{cases} \tag{3.14}$$

Let

$$\lambda \sup_{v_j} |f'(v_j)| \leq \frac{1}{2}. \tag{3.15}$$

The condition like (3.15) is called Courant-Friedrichs-Lewy(CFL) condition. Then waves from two neighbouring Riemann problems do not intersect before time Δt. Infact (3.15) ensures that a wave from $x_{i+\frac{1}{2}}$ will not reach the line$x = x_i$ and $x = x_{i+1}$ before time Δt. Hence problem (3.13) can be seen as a **superposition of local Riemann problems** under the condition (3.15). The solution of (3.13) in $(x_i, x_{i+1}) \times (t_n, t_{n+1}]$ is given by

$$w(x,t) = w_R((x - x_{i+\frac{1}{2}})/t - t_n, v_i^n, v_{i+1}^n), x \in (x_i, x_{i+1}), t \in (t_n, t_{n+1}].$$

Now solution at $t = t_{n+1}$ is given by

$$w(x,t_{n+1}) = w_R((x - x_{i+\frac{1}{2}})/\Delta t, v_i^n, v_{i+1}^n), x \in (x_i, x_{i+1})$$

Step 2: Define v_i^{n+1} by

$$\begin{aligned} v_i^{n+1} &= \frac{1}{h}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} w(x,t_{n+1})\,dx \\ &= \int_{x_{i-\frac{1}{2}}}^{x_i} w_R((x - x_{i-\frac{1}{2}})/\Delta t, v_{i-1}^n, v_i^n)\,dx + \int_{x_i}^{x_{i+\frac{1}{2}}} w_R((x - x_{i+\frac{1}{2}})/\Delta t, v_i^n, v_{i+1}^n)\,dx \\ &= \int_{\frac{-h}{2}}^{0} w_R(x/\Delta t, v_{i-1}^n, v_i^n)\,dx + \int_0^{\frac{h}{2}} w_R(x/\Delta t, v_i^n, v_{i+1}^n)\,dx \end{aligned}$$

In order to get a simple expression for v_i^{n+1}, integrate the equation (3.13) over the cell $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (t_n, t_{n+1})$,then we have

$$\begin{aligned} 0 &= \int_{t_n}^{t_{n+1}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} (w_t + f(w)_x)\,dx\,dt \\ &= \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} w(x,t_{n+1})\,dx - \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} w(x,t_n)\,dx \\ &\quad + \int_{t_n}^{t_{n+1}} f(w(x_{i+\frac{1}{2}} - 0, t)\,dt + \int_{t_n}^{t_{n+1}} f(w(x_{i-\frac{1}{2}} + 0, t)\,dt \\ &= h(v_i^{n+1} - v_i^n) + \Delta t(f(w_R(0-, v_i^n, v_{i+1}^n)) - f(w_R(0+, v_{i-1}^n, v_i^n))) \end{aligned}$$

But by (**??**),

$$f(w_R(0-, v_i^n, v_{i+1}^n)) = f(w_R(0+, v_i^n, v_{i+1}^n)) = f(w_R(0, v_i^n, v_{i+1}^n))$$

Now **Godunov scheme** is given by

$$v_i^{n+1} = v_i^n - \lambda(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n) \tag{3.16}$$

where its numerical flux

$$F_{i+\frac{1}{2}}^n = F^G(v_i^n, v_{i+1}^n) = f(w_R(0, v_i^n, v_{i+1}^n)).$$

Godunov has given the following simple formula for a general f to evaluate the numerical flux

$$\begin{aligned} F^G(u,v) &= f(w_R(0,u,v)) \\ &= \begin{cases} \min_{w\in[u,v]} f(w) & \text{if} \quad u \leq v \\ \max_{w\in[v,u]} f(w) & \text{if} \quad u \geq v \end{cases} \end{aligned} \tag{3.17}$$

Godunov scheme (3.16) is in **conservative form, consistent** with the conservation law (3.1) and also an **upwind scheme** i.e.,

$$F^G(u,v) = \begin{cases} f(u) \text{ if } f \text{is a nondecreasing function between } u \text{ and } v \\ f(v) \text{ if } f \text{is a nonincreasing function between } u \text{ and } v \end{cases}$$

Remark: If f is a convex function and $f(\theta) = \min_{w\in I\!R} f(w)$,then the Godunov flux (3.17) can be expressed in more simpler form

$$F^G(u,v) = \max(f(\max(u,\theta)), f(\min(v,\theta))).$$

Similarly if f is concave and $f(\theta) = \max_{w\in I\!R} f(w)$,then

$$F^G(u,v) = \min(f(\min(u,\theta)), f(\max(v,\theta))).$$

Remark: As Godunov flux $F_{i+\frac{1}{2}}^n$depends on the solution of Riemann problem at $x = x_{i+12}$ only and the waves from $x = x_{i+\frac{1}{2}}$ does not reach $x = x_{i-\frac{1}{2}}$ and $x = x_{i+\frac{3}{2}}$ in time Δt, we can relax the codition (3.15) by

$$\lambda \sup_{v_j} |f'(v_j)| \leq 1 \tag{3.18}$$

Example 3.2. The Lax-Friedrichs scheme:

This is the simplest centered finite scheme which is given by

$$\begin{aligned} v_i^{n+1} &= v_i^n - \frac{\lambda}{2}(f(v_{i+1}^n) - f(v_{i-1}^n)) \\ &= v_i^n - \lambda(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n) \end{aligned} \tag{3.19}$$

where

$$F_{i+\frac{1}{2}}^n = F^{LF}(v_i^n, v_{i+1}^n) = \frac{1}{2}(f(v_i^n) + f(v_{i+1}^n) - \frac{1}{\lambda}(v_{i+1}^n - v_i^n))$$

This is in a conservative form and is consistent with conservation law (3.1) but is not an upwind scheme. However this scheme can be interpreted as a projection of the solution of successive non-interacting Riemann problems. Let $w(x,t) = w_R(\frac{x-x_i}{t}, v_{i-1}^n, v_{i+1}^n)$ be the solution of Riemann problem

$$\begin{cases} w_t + f(w)_x = 0, \ \ t \in (t_n, t_{n+1}] \\ w(x, t_n) = \begin{cases} v_{i-1}^n & \text{if} \quad x < x_i \\ v_{i+1}^n & \text{if} \quad x > x_i \end{cases} \end{cases} \tag{3.20}$$

where

$$x_i = \frac{x_{i+\frac{1}{2}} + x_{i-\frac{1}{2}}}{2} \ \forall i \in Z.$$

Here we consider Riemann problems centered at $x = ..., x_{i-4}, x_{i-2}, x_i, x_{i+2i}, ...$ so that the waves from nighbouring cells does not interact. Observe that under the CFL condition (3.15),the solution of (3.20) $w(x,t)$ satisfies, $w(x_{i-1}+0,t) = v_{i-1}^n$ and $w(x_{i+1}-0,t) = v_{i+1}^n \ \forall t \in (t_n, t_{n+1})$. Define

$$v_i^{n+1} = \int_{x_{i-1}}^{x_{i+1}} w_R(\frac{x-x_i}{\Delta t}, v_{i-1}^n, v_{i+1}^n)$$

Like in the derivation of Godunove scheme,by integrating over the rectangle $(x_{i-1}, x_{i+1}) \times (t_n, t_{n+1})$ we have

$$\begin{aligned} 0 &= \int_{t_n}^{t_{n+1}} \int_{x_{i-1}}^{x_{i+1}} (w_t + f(w)_x)\,dx\,dt \\ &= \int_{x_{i-1}}^{x_{i+1}} w(x, t_{n+1})\,dx - \int_{x_{i-1}}^{x_{i+1}} w(x, t_n)\,dx \\ &\quad + \int_{t_n}^{t_{n+1}} f(w(x_{i+1}-0,t)\,dt + \int_{t_n}^{t_{n+1}} f(w(x_{i-1}+0,t)\,dt \\ &= 2h(v_i^{n+1} - \frac{v_{i-1}^n + v_{i+1}^n}{2}) + \Delta t(f(v_{i+1}^n)) - f(v_{i-1}^n))) \end{aligned}$$

This is Lax-Friedrichs scheme.

Example 3.3. The Murman-Roe scheme:

Define

$$a(u,v) = \begin{cases} \frac{(f(u)-f(v))}{u-v} & \text{if } u \neq v \\ f'(u) & \text{if } u = v \end{cases} \tag{3.21}$$

We proceed as in the Godunov scheme,only replacing the local Riemanproblem (3.14) by that of linear problem

$$\begin{cases} w_t + a(v_i^n, v_{i+1}^n)w_x = 0, \\ w(x,t_n) = \begin{cases} v_i^n & \text{if} \quad x < x_{i+\frac{1}{2}} \\ v_{i+1}^n & \text{if} \quad x > x_{i+\frac{1}{2}} \end{cases} \end{cases} \tag{3.22}$$

The solution of (3.22),

$$w(x,t) = w_R^{Roe}\left(\frac{x - x_{i+\frac{1}{2}}}{t}, v_i^n, v_{i+1}^n\right) = \begin{cases} v_i^n & \text{if } x - x_{i+\frac{1}{2}} < a(v_i^n, v_{i+1}^n)t \\ v_{i+1}^n & \text{if } x - x_{i+\frac{1}{2}} > a(v_i^n, v_{i+1}^n)t \end{cases} \tag{3.23}$$

has a discontinuity moving with a speed $a(v_i^n, v_{i+1}^n)$.

Exactly like in the construction of Godunov scheme,we have **Murman-Roe** scheme

$$v_i^{n+1} = v_i^n - \lambda(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n) \tag{3.24}$$

where

$$F_{i+\frac{1}{2}}^n = F^{Roe}(v_i^n, v_{i+1}^n) = f(w_R^{Roe}(0, v_i^n, v_{i+1}^n)).$$

This flux can be written further in the following form

$$F^{Roe}(u,v) = \frac{1}{2}(f(u) + f(v) - |a(u,v)|(v-u))$$

Murman-Roe scheme is in **conservative form,consistent** with the conservation law (3.1) and also an **upwind scheme**.

Example 3.4. The Engquist-Osher scheme:

Engquist-Osher scheme is given by

$$v_i^{n+1} = v_i^n - \lambda(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n) \tag{3.25}$$

where the numerical flux $F_{i+\frac{1}{2}}^n = F^{EO}(v_i^n, v_{i+1}^n)$ and

$$F^{EO}(u,v) = \frac{1}{2}(f(u) + f(v) - \int_u^v |f'(\xi)|\, d\xi) \tag{3.26}$$

Enguist-Osher scheme is in **conservative form,consistent** with the conservation law (3.1) and also an **upwind scheme**.

Remark: if f is a convex function and $f(\theta) = \min_{w\in I\!R} f(w)$,then the Engquist-Osher flux (3.26) can be expressed in more simpler form

$$F^{EO}(u,v) = f(\max(u,\theta)) + f(\min(v,\theta)) - f(\theta)$$

. Similarly if f is concave and $f(\theta) = \max_{w\in I\!R} f(w)$,then

$$F^{EO}(u,v) = f(\min(u,\theta)) + f(\max(v,\theta)) - f(\theta)$$

.

Example 3.5. The Lax-Wendroff scheme:

This scheme is derived by a Taylor series expansion of a smooth solution of (3.1),where the term of order strictly greater than two are neglected,thus leading to a second order scheme. Let u be a smooth solution of (3.1),then we have

$$\begin{aligned} u(x,t+\Delta t) &= u(x,t) + \Delta t\ u_t(x,t) + (\Delta t^2/2)\, u_{tt}(x,t) + 0(\Delta t^3) \\ &= u(x,t) - \Delta t\ f(u)_x + (\Delta t^2/2)\ + (f'(u)(f(u)_x)_x + 0(\Delta t^3) \end{aligned}$$

Now

$$\begin{aligned} f(u)_x &= \frac{f(u(x+h,t)) - f(u(x-h,t))}{2h} + o(h^2) \\ (f'(u)(f((u)_x)_x &= [f'((u(x+h,t) + u(x,t))/2)(f(u(x+h,t) - f(u(x,t)))- \\ & \qquad f'((u(x,t) + u(x-h,t))/2)(f(u(x,t) - f(u(x-h,t)))]/h^2\ + o(h). \end{aligned}$$

This leads to the following **Lax-Wendroff** scheme

$$v_i^{n+1} = v_i^n - \lambda(F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}}) \tag{3.27}$$

where the numerical flux $F^n_{i+\frac{1}{2}} = F^{LW}(v_i^n, v_{i+1}^n)$ and

$$F^{LW}(u,v) = \frac{1}{2}(f(u) + f(v) - \lambda\, f'(\frac{u+v}{2})(f(v) - f(u))) \tag{3.28}$$

Lax-Wendroff scheme is in **conservative form,consistent** with the conservation law (3.1) and but **not an upwind scheme**

3.1 Monotone and TVD Schemes

<u>Definition</u>: The finite difference scheme (3.2) is said to be **monotone** if H is non decreasing function of each of its arguments.

Notation: Let $v = (v_i)_{i \in Z}$ and $w = (w_i)_{i \in Z}$ then $v \geq w$ means $v_i \geq w_i \; \forall i \in Z$.

Let $(H_\Delta(v))_i = H(v_{i-k}, v_{i-k+1}, ..., v_i, ..., v_{i+k-1}, v_{i+k})$ and $H_\Delta(v) = (H_\Delta(v))_i)_{i \in Z}$ then scheme (3.2) is monotone means,

$$v \geq w \; \Rightarrow \; H_\Delta(v) \geq H_\Delta(w)$$

Examples of monotone schemes: Godunov,Lax-Friedrichs and Enquist-Osher schemes are monotone under the CFL like condition

$$\lambda \sup_{v_j} |f'(v_j)| \leq 1. \tag{3.29}$$

Proposition 3.2: If a 3-point scheme which is in conservative form and is monotone,then the numerical flux $F(u, v)$ is an incrasing function in its first argumennt,and a decreasing function in its second argument.

Proof: By the definition of a 3-point conservative scheme,

$$H(v_{-1}, v_0, v_1) = v_0 - \lambda(F(v_0, v_1) - F(v_{-1}, v_0)).$$

Thus the mapping

$$u \rightarrow F(u, v_0) = \frac{1}{\lambda}[H(u, v_0, v_1) - v_0] + F(v_0, v_1)$$

is increasing iff $u \rightarrow H(u, v_0, v_1)$ is increasing and

$$v \rightarrow F(v_0, v) = -\frac{1}{\lambda}[H(v_{-1}, v_0, v) - v_0] + F(v_{-1}, v_0)$$

is decreasing iff $u \rightarrow H(v_{-1}, v_0, v)$ is increasing.

Converse of Proposition 3.2:
Suppose F(u,v) is increasing in its first argument and decreasing in second argument,then the scheme is monotone provided,

$$\lambda \max_{w,z}\{|F(u, w) - F(v, w)| + |F(z, u) - F(z, v)|\} \leq |u - v| \; \forall u, v.$$

Given a sequence $v = (v_i)_{i \in Z}$ we define the following norms

$$||v||_1 = h \sum_{i \in Z} |v_i| \tag{3.30}$$

$$||v||_\infty = \max_{i\in Z} |v_i| \tag{3.31}$$

$$TV(v) = \sum_{i\in Z} |v_{i+1} - v_i| \tag{3.32}$$

Lemma 3.3. (Crandall-Tartar): Let (Ω, μ) be a mesure space and C be a subset of $L^1(\Omega)$ such that

$$f, g \in C \Rightarrow \max(f, g) \in C$$

Let T be a mapping from C to $L^1(\Omega)$ which satisfies

$$\int_\Omega T(f)\, d\mu = \int_\Omega f\, d\mu \;\; \forall f \in C. \tag{3.33}$$

Then the following properties are equivalent:
(a) T is order preserving i.e.,

$$f, g \;\in C \;\text{ and }\; f \leq g\; a.e \Rightarrow T(f) \leq T(g) \;\; a.e$$

(b)

$$\int_\Omega |T(f) - T(g)|_+ \, d\mu \;\leq\; \int_\Omega |f - g|_+ \, d\mu \quad \forall f, g \in C.$$

(c)

$$\int_\Omega |T(f) - T(g)| \, d\mu \;\leq\; \int_\Omega |f - g| \, d\mu \quad \forall f, g \in C.$$

Proof: (a) $\Rightarrow$ (b):
Since $\max(f, g) = g + (f - g)_+$,by the property (a) we have

$$T(\max(f, g)) \;\geq\; T(g)$$

and similarly

$$T(\max(f, g)) \;\geq\; T(f)$$

By subtracting $T(g)$ from both the sides of above inequalities,we get

$$T(\max(f, g)) - T(g) \;\geq\; (T(f) - T(g))_+$$

Integrate over Ω the above inequality,we get

$$\begin{aligned}\int_\Omega (T(f) - T(g))_+ \, d\mu \;&\leq\; \int_\Omega (T(\max(f, g)) - T(g))\, d\mu \\ &=\; \int_\Omega (\max(f, g) - g)\, d\mu \;\text{ by }\; (3.33)\end{aligned}$$

$$= \int_\Omega (f-g)_+ \, d\mu$$

(b) $\Rightarrow$ (c):
Using the identity

$$|f-g| = (f-g)_+ \; + \; (g-f)_+$$

we have

$$\begin{aligned} \int_\Omega |T(f)-T(g)| \, d\mu &= \int_\Omega (T(f)-T(g))_+ \, d\mu \; + \int_\Omega (T(g)-T(f))_+ \, d\mu \\ &\leq \int_\Omega (f-g)_+ \, d\mu \; + \int_\Omega (g-f))_+ \, d\mu \\ &= \int_\Omega |f-g| \, d\mu \end{aligned}$$

(c) $\Rightarrow$ (a):
Since $(f-g)_+ \; = \; \frac{1}{2}(|f-g| + (f-g))$ and by the property (3.33),we get

$$\begin{aligned} \int_\Omega (T(f)-T(g))_+ \, d\mu &= \frac{1}{2} \int_\Omega (|T(f)-T(g)| \quad + \quad (T(f)-T(g))) \, d\mu \\ &\leq \frac{1}{2} \int_\Omega (|f-g| \quad + \quad (f-g)) \, d\mu \\ &= \int_\Omega (f-g)_+ \end{aligned}$$

Suppose $f \leq g$ a.e,then $(f-g)_+ = 0$ a.e,hence we get

$$\int_\Omega (T(f)-T(g))_+ \, d\mu \; \leq \; 0$$

which implies $T(f) \leq T(g)$ a.e. This completes the proof of Lemma 3.3.

Lemma 3.4.(l^∞ stability): Let $v^n = (v_i^n)_{i \in Z} \in l^\infty$ and scheme (3.4) is monotone, consistent,then $v^{n+1} = (v_i^{n+1})_{i \in Z} \in l^\infty$ and

$$||v^{n+1}||_\infty \leq ||v^n||_\infty \tag{3.34}$$

Proof: Since H is increasing in each of its arguments,we have

$$\min_{i-k \leq j \leq i+k} \{v_j^n\} \leq H(v_{i-k}^n, ..., v_{i+k}^n) \leq \max_{i-k \leq j \leq i+k} \{v_j^n\}$$

. Let

$$|v_m| = \max_{i-k \leq l \leq i+k} \{|v_l^n|\}.$$

Then by consisstency and monotonicity of the scheme we, have

$$-|v_m| = H(-|v_m|, -|v_m|, ..., -|v_m|) \leq H(v_{i-k}^n, ..., v_{i+k}^n) \leq H(|v_m|, |v_m|, ..., |v_m|) = |v_m|.$$

Hence we have

$$|v_i^{n+1}| \leq |v_m| = \max_{i-k \leq l \leq i+k} \{|v_l^n|\}. \tag{3.35}$$

This implies the result.

Let

$$v^{n+1} = H_\Delta(v^n) = ((H_\Delta(v^n))_i)_{i \in Z}$$

and

$$(H_\Delta(v^n))_i = H(v_{i-k}^n, ..., v_{i+k}^n) = v_i^{n+1}.$$

Lemma 3.5(l^1 contraction) Suppose $v^n \in l^\infty$ and the scheme (3.2) is in conservative form,consistent and monotone. Then $H_\Delta : l^1 \to l^1$ is a mapping which preserves the integral and for any sequence u and v in l^1 we have

$$||H_\Delta(u) - H_\Delta(v)||_1 \leq ||u - v||_1 \tag{3.36}$$

Proof:(a) From (3.35) we have

$$|(H_\Delta(v^n))_i| \leq \max_{i-k \leq l \leq i+k} \{|v_l^n|\} \leq \sum_{l=i-k}^{i+k} |v_l^n|.$$

By taking summation over i and multiplying by h we get

$$h \sum_{i \in Z} |(H_\Delta(v^n))_i| \leq \leq (2k+1)h \sum_{i \in Z} |v_l^n|.$$

$$\Rightarrow v^{n+1} \in l^1$$

.

(b) By taking summation over i and multiplying by h in (3.4) we get

$$h \sum_{i \in Z} v_i^{n+1} = h \sum_{i \in Z} v_i^n.$$

Hence H_Δ preserves the integral.

(c) H_Δ is order preserving follows from scheme is monotone. Hence by Crandall and Tartar lemma we have

$$||H_\Delta(u) - H_\Delta(v)||_1 \leq ||u - v||_1$$

i.e.,

$$h\sum_{i\in Z}|u_i^{n+1}-v_i^{n+1}| = h\sum_{i\in Z}|u_i^n - v_i^n|.$$

<u>Definition:</u> A scheme is said to be **total variation diminishing(TVD)** if

$$TV(v^{n+1}) \leq TV(v^n)\ \forall n = 0,1,2,....$$

Lemma 3.6(TVD): Any conservative,consistent monotone scheme satisfies total variation diminishing property i.e.,

$$\begin{aligned}\sum_{i\in Z}|v_i^{n+1}-v_{1-1}^{n+1}| &= \sum_{i\in Z}|v_i^n - v_{i-1}^n|. \\ i.e.,\quad Tv(v^{n+1}) &\leq TV(v^n) \end{aligned} \tag{3.37}$$

Proof: By taking $u_i^n = v_{i-1}^n$ in (3.36),we get (3.37).

Lemma 3,7.(Time estimate): Let scheme (3.2) be in conservative form, consistent with (3.1) and TVD with a Lipschitz continuous numerical flux $F_{i+\frac{1}{2}}$. Assume moreover that scheme is l^∞ stable. Then there exists a constant $C > 0$ such that $\forall\ 0\ \leq m \leq n$,

$$h\sum_{i\in Z}|v_i^m - v_i^n| \ \leq\ C(m-n)\Delta t TV(v^0) \tag{3.38}$$

Proof: From (3.4) we have,

$$v_i^{n+1} - v_i^n = \lambda(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}})$$

and since F is lipschitz continuous,there exists a constant $c_1 = c_1(F, ||v^0||_\infty) > 0$ such that for $C = 2kc_1$, $n > m$,

$$\begin{aligned}|v_i^n - v_i^{n-1}| &\leq \lambda c_1\{|v_{i-k+1}^{n-1} - v_{i-k}^{n-1}| + ..., |v_{i+k}^{n-1} - v_{i+k-1}^{n-1}| \\ &\leq 2kc_1 TV(v^{n-1}) \\ &\leq CTV(v^m)\end{aligned}$$

Since scheme is TVD and

$$|v_i^n - v_i^m| \leq |v_i^n - v_i^{n-1}| + |v_i^{n-1} - v_i^{n-2}| + ... + |v_i^{m+1} - v_i^m|$$

then we get by taking summation over i and multiplying by h,

$$h\sum_{i\in Z}|v_i^n - v_i^m| \ \leq\ C(m-n)\Delta t TV(v^m).$$

This completes the proof.

Let v_h defined by (3.7) be the piecewise constant function i.e.,

$$v_h(x,t) = v_i^n \text{ for } (x,t) \in (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times [t_n, t_{n+1}) \tag{3.39}$$

Let $0 \leq s \leq t \leq T$ for any $T > 0$. Let t_n, t_m be such that $t_m \leq s < t_{m+1}$ and $t_n \leq t < t_{n+1}$. Then

$$|t_n - t_m| \leq |t - s| + \Delta t$$

and

$$v_h(.,t) - v_h(.,s) = v_h(.,t_n) - v_h(.,t_m)$$

Then under the assumption of Lemma 3.7 we have

$$||v_h(.,t) - v_h(.,s)||_{L^1} \leq C(|t-s| + \Delta t)TV(v_h(.,0) \tag{3.40}$$

for some constant $C > 0$, $0 \leq s \leq t \leq T$. Here C is independent of t. Equation (3.40) says that v_h is **Lipschitz continuous in** t w.r.t L^1 norm.

Now we state the final result for convergence.

Theorem 3.8(Existence of a weak solution): Let for any $T > 0$,
(a) $u_0 \in L^\infty(I\!R)$ and $BV(I\!R)$,a functions of bounded variations and v^0 given by (3.3).
(b) $v_h(x,t)$ be an approximate solution obtained from a difference scheme (3.2) which is in conservative form and consistent with (3.1).
(c) $||v_h(.,t)||_\infty < \infty$ and a $TV(v_h(.,t)) < \infty$ for $0 \leq t \leq T$.
(d)

$$||v_h(.,t) - v_h(.,s)||_{L^1} \leq C(|t-s| + \Delta t)TV(v_h(.,0)$$

for some constant $C > 0, 0 \leq s \leq t \leq T$.

Then there exists a sequence $h_k \to 0$ such that if we set,$\Delta_k t = \lambda h$,with λ being kept constant,the sequence v_{h_k} converges in $L^\infty(0,T; L^1_{loc}(I\!R))$, say to u. This limit u is a weak solution of (3.1).

Proof: Let Ω be a bounded openset in $I\!R$. By (a) and (b) $v_h(.,t) \in BV(\Omega)$ as well as in $L^1(\Omega)$ for any $0 \leq t \leq T$.

Let $(s_n)_{n=0}^\infty$ be a countable dense subset of $[0,T]$. Since the canonical embedding $BV(\Omega)$ in $L^1(\Omega)$ is compact for any bounded set of Ω, then for each s_k there exists a subsequence h_k of $h \to 0$ such that $v_{h_k}(\cdot, s_k)$ coverges ,say to $u(\cdot, s_k)$ in $L^1_{loc}(I\!R)$. Again by a diagonalisation argument we find another subsequence of h_k still denoted by h_k such that $v_{h_k}(\cdot, s_j) \to u(\cdot, s_j)$ for all j. In order to reach every point t in $[0, T]$,we partition the interval $[0, T]$ into N intervals (t_i, t_{i+1}) where t_i are selected from the dense set $(s_n)_{n=0}^{\infty}$ in such a way that each interval has length less than $\epsilon > 0$, where ϵ is any given arbitrary positive constant. For any $t \in [0, T]$ there exists t_i such that $|t - t_i| \leq \epsilon$ and

$$\begin{aligned} ||v_{h_k}(\cdot, t) - v_{h_m}(\cdot, t)||_{L^1(\Omega)} \leq \; & ||v_{h_k}(\cdot, t) - v_{h_k}(\cdot, t_i)||_{L^1(\Omega)} + ||v_{h_k}(\cdot, t_i) - v_{h_m}(\cdot, t_i)||_{L^1(\Omega)} + \\ & ||v_{h_m}(\cdot, t_i) - v_{h_m}(\cdot, t)||_{L^1(\Omega)} \end{aligned}$$

By (d),there exists a constant C independent of t such that first and the last term are bounded by $C(\Delta t + \epsilon)$. The term in the middle can be made less than ϵ by the fact that $v_{h_k}(\cdot, s_j) \to u(\cdot, s_j)$ for all j in $L^1_{loc}(I\!R)$. Hence $u_{h_k}(\cdot, t)$ is a uniform in $t \in [0, T]$ cauchy sequence in $L^1_{loc}(I\!R)$. Therefore we can extract a subsequence $h_k \to 0$ such taht v_{h_k} converges in $L^\infty(0, T; L^1_{\rm loc}(R))$ say to $u(\cdot, t)$ and almost everywhere. From this we can conclude that there exists a subsequence $h_k \to 0$ such that v_{h_k} converges in $L^1_{loc}(I\!R \times (0, \infty))$ and a.e to a function u. By Lax-Wendroff's theorem u is a weak solution of conservation law (3.1). This completes the proof of theorem.

Descretized version of Kruzkov's entropy condition (1.23):

Now we show that monotone scheme satisfies the descretized version of the Kruzkov entropy condition:

$$|u_i^{n+1} - l| - |u_i^n - l| + \lambda(G_{i+1/2}^n - G_{i-1/2}^n) \leq 0$$

for any real number l, where $G_{i+1/2}^n = G(v_{i-k+1}^n, ..., v_i^n, ..., v_{i+k}^n)$ is such that

$$G(u, u) = sgn(u - l)(f(u) - f(l)). \tag{3.41}$$

Consider a monotone scheme which is in conservation form and consistent with (3.1):

$$\begin{aligned} v_i^{n+1} &= H(v_{i-k}^n, ..., v_i^n, ..., v_{i+k}^n) \\ & \quad v_i^n - \lambda(F_{i+1/2}^n - F_{i-1/2}^n) \end{aligned} \tag{3.42}$$

where H is non-decreasing in each of its arguments and $F_{i+1/2}^n = F(v_{i-k+1}^n, ..., v_i^n, ..., v_{i+k}^n)$ with $F(v, v, ..., v, v) = f(v)$. Set $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$. Define

$$G(v_{i-k+1}^n, ..., v_{i+k}^n) = F(v_{i-k+1}^n \vee l, ..., v_{i+k}^n \vee l) - F(v_{i-k+1}^n \wedge l, ..., v_{i+k}^n \wedge l).$$

Then $G(v, ..., v) = sgn(v - l)(f(v) - f(l))$ satisfies the condition (3.41).
Consider

$$\begin{aligned} -\lambda(G_{i+1/2} - G_{i-1/2}) &= -\lambda(F(v_{i-k+1} \vee l, ..., v_{i+k} \vee l) - F(v_{i-k} \vee l, v_{i+k-1} \vee l)) \\ & \quad +\lambda(F(v_{i-k+1} \wedge l, ..., v_{i+k}, \wedge l) - F(v_{i-k} \wedge l, ..., v_{i+k-1} \wedge l)) \\ &= H(v_{i-k} \vee l, ..., v_i \vee l, ..., v_{i+k} \vee l) - v_i \vee l \\ & \quad -H(v_{i-k} \wedge l, ..., v_i \wedge l, ..., v_{i+k} \wedge l) + v_i \wedge l \end{aligned}$$

$\Rightarrow$

$$|v_i - l| - \lambda(G_{i+1/2} - G_{i-1/2}) = H(v_{i-k} \vee l, ..., v_i \vee l, ..., v_{i+k} \vee l) - H(v_{i-k} \wedge l, ..., v_i \wedge l, v_{i+k} \wedge l).$$

Note that

$$H(v_{i-k}^n \vee l, ..., v_i^n \vee l, ..., v_{i+k}^n \vee l) \quad \geq \quad H(v_{i-k}^n, ..., v_i^n, ..., v_{i+k}^n) \vee H(l, l, l)$$

$$= \quad v_i^{n+1} \vee l$$

Similarly

$$\begin{aligned} H(v_{i-k}^n \wedge l, ..., v_i^n \wedge l, ..., v_{i+k}^n \wedge l) &\leq H(v_{i-k}^n, ..., v_i^n, ..., v_{i+k}^n) \wedge H(l,l,l) \\ &= v_i^{n+1} \wedge l \end{aligned}$$

$\Rightarrow$

$$|v_i^{n+1} - l| - |v_i^n - l| + \lambda(G_{i+1/2}^n - G_{i-1/2}^n) \leq 0$$

Corollary 3.9.: Solution obtained from a consistent,conservative monotone scheme converges to the entropy solution of (3.1).

Proof: solution obtained from any consistent,conservative monotone scheme satisfies the hypothesis of theorem 3.8, hence there exists a subsequence h_k such that v_{h_k} converges in $L^\infty(0,T;L^1_{loc}(I\!R))$ say to u. Since approximate solution obtained from monotone scheme satisfies the descrete entropy condition,by uniqueness, all the subsequence of v_{h_k} converges to u only. This implies whole sequence v_h converges to u. This u is a weak solution and satisfies the entropy condition.

Example of scheme which converges to weak solution but not to entropy solution:

Consider a Murman-Roe scheme for the following Riemann problem:

$$\begin{cases} u_t + (\frac{u^2}{2})_x = 0 \text{ in } I\!R \times (0,\infty) \\ u(x,0) = \begin{cases} -1 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x > 0 \end{cases} \end{cases} \tag{3.43}$$

Then we have $v_i^{n+1} = v_i^n \quad \forall i \in Z$. This scheme admits entropy violating stationary shocks. Hence this is not a monotone scheme.

One of the important criteria to establish the convergence of the scheme is total variation bound. In the next section we give general criteria which ensures the scheme is TVD. Monotone schemes obviously satisfies this criteria.

3.2 Incremental form and Numerical viscosity

Definition : A scheme is said to be in an **incremental form** if there exists functions C, D called incremental coefficients such that

$$v_i^{n+1} = v_i^n - C_{i-\frac{1}{2}}\Delta v_{i-\frac{1}{2}} + D_{i+\frac{1}{2}}\Delta v_{i+\frac{1}{2}} \tag{3.44}$$

where $\Delta v_{i+\frac{1}{2}} = v_{i+1} - v_i$, $C_{i+\frac{1}{2}} = C(v_i^n, v_{i+1}^n)$ and $D_{i+\frac{1}{2}} = D(v_i^n, v_{i+1}^n)$.

Example: Any consistent,conservative scheme (3.4) with Lipschitz continuous flux admits a unique incremental form with incremental coefficients given by

$$\begin{aligned} C^n_{i+\frac{1}{2}} &= \lambda(f(v^n_{i+1}) - F^n_{i+\frac{1}{2}})/\Delta v^n_{i+\frac{1}{2}} \\ \text{and} & \\ D^n_{i+\frac{1}{2}} &= \lambda(f(v^n_{i}) - F^n_{i+\frac{1}{2}})/\Delta v^n_{i+\frac{1}{2}} \end{aligned} \tag{3.45}$$

where $F^n_{i+\frac{1}{2}} = F(v_i^n, v_{i+1}^n)$.

Definition : A scheme is said to be in **viscous form** if there exists a function Q called the coefficient of numerical viscosity such that if we set $Q^n_{i+\frac{1}{2}} = Q(v_i^n, v_{i+1}^n)$ the scheme can be written

$$v_i^{n+1} = v_i^n - \frac{\lambda}{2}(f(v^n_{i+1}) - f(v^n_{i-1})) + \frac{1}{2}(Q^n_{i+\frac{1}{2}}\Delta v^n_{i+\frac{1}{2}} - Q^n_{i-\frac{1}{2}}\Delta v^n_{i-\frac{1}{2}}). \tag{3.46}$$

Any 3-point scheme with Lipschitz numerical flux admits a unique viscous form whose numerical viscosity coefficient is given by

$$Q_{i+\frac{1}{2}} = \lambda(f(v_i) + f(v_{i+1}) - 2F_{i+\frac{1}{2}})/\Delta v_{i+\frac{1}{2}}$$

.

Moreover,

$$Q_{i+\frac{1}{2}} = C_{i+\frac{1}{2}} + D_{i+\frac{1}{2}}$$

$$C_{i+\frac{1}{2}} = \frac{\lambda}{2}(\frac{f(v_{i+1}) - f(v_i)}{v_{i+1} - v_i}) + \frac{Q_{i+\frac{1}{2}}}{2}$$

and

$$D_{i+\frac{1}{2}} = \frac{-\lambda}{2}(\frac{f(v_{i+1}) - f(v_i)}{v_{i+1} - v_i}) + \frac{Q_{i+\frac{1}{2}}}{2}.$$

where $C_{i+\frac{1}{2}}, D_{i+\frac{1}{2}}$ are the unique incremental coefficients defined by (3.45). Note that $C_{i+\frac{1}{2}} \geq 0$ and $D_{i+\frac{1}{2}} \geq 0$ iff $Q_{i+\frac{1}{2}} \geq \lambda |\frac{f(v_{i+1})-f(v_i)}{v_{i+1}-v_i}|$

For example,Coefficient of numerical viscosty of Lax-Friedrichs,Enguist-Osherand Murman-Roe schemes are given by
(a) Lax-Friedrichs scheme,$Q_{i+\frac{1}{2}} = Q^{LF}_{i+\frac{1}{2}} = 1$,
(b) Engquist-Osher scheme, $Q_{i+\frac{1}{2}} = Q^{EO}_{i+\frac{1}{2}} = \lambda \int_{v_i^n}^{v_{i+1}^n} |f'(\xi)| \, d\xi / \Delta v_{i+\frac{1}{2}}$,
(c) Murman-Roe scheme $Q_{i+\frac{1}{2}} = Q^{Roe}_{i+\frac{1}{2}} = \lambda |\frac{f(v_{i+1})-f(v_i)}{v_{i+1}-v_i}|$.

Exerscise: Show that $Q^{Roe}_{i+\frac{1}{2}} \leq Q^{Godunov}_{i+\frac{1}{2}} \leq Q^{EO}_{i+\frac{1}{2}} \leq Q^{LF}_{i+\frac{1}{2}} = 1$ under the CFL like condition (3.29).

We now give a very useful creteria due to Harten which ensures that a scheme in incremenatal form is TVD.

Lemma(Harten): Let (3.2) be a difference scheme which is in incremental form (3.44). Assume that incremental coefficients satisfy the following conditions

$$C^n_{i+\frac{1}{2}} \geq 0, \quad D^n_{i+\frac{1}{2}} \geq 0 \ \forall i \tag{3.47}$$

$$C^n_{i+\frac{1}{2}} + D^n_{i+\frac{1}{2}} \leq 1 \ \forall i \tag{3.48}$$

Then the scheme is TVD.

Proof:

$$v_i^{n+1} - v_{i-1}^{n+1} = (1 - C^n_{i-\frac{1}{2}} - D^n_{i-\frac{1}{2}})(v_i^n - v_{i-1}^n) + C^n_{i-\frac{3}{2}}(v_{i-1}^n - v_{i-2}^n) + D^n_{i+\frac{1}{2}}(v_{i+1}^n - v_i^n)$$

By assumption the coefficients of the right hand side are positive,hence we have

$$|v_i^{n+1} - v_{i-1}^{n+1}| \leq (1 - C^n_{i-\frac{1}{2}} - D^n_{i-\frac{1}{2}})|v_i^n - v_{i-1}^n| + C^n_{i-\frac{3}{2}}|v_{i-1}^n - v_{i-2}^n| + D^n_{i+\frac{1}{2}}|v_{i+1}^n - v_i^n|$$

By taking summation over $i \in Z$ and rearranging the sum, we have

$$\sum_{i \in Z} |v_i^{n+1} - v_{i-1}^{n+1}| \leq \sum_{i \in Z} |v_i^n - v_{i-1}^n|.$$

This completes the proof.

From the relations between $Q_{i+\frac{1}{2}}$, $C^n_{i+\frac{1}{2}}$ and $D^n_{i+\frac{1}{2}}$ we have the following result.

Corollary: Let (3.2) be a difference scheme which can be put in viscous form(3.46). Assume that numerical viscosity coefficient satisfies

$$\lambda|\frac{f(v_{i+1}) - f(v_i)}{v_{i+1} - v_i}| \leq Q_{i+\frac{1}{2}} \leq 1. \tag{3.49}$$

Then the scheme is TVD.

Murman-Roe Scheme and its Entropy modification: Since Murman-Roe scheme admits entropy violating stationary shocks,Harten-Hyman have proposed the following modification. Define, for a given $\epsilon > 0$

$$Q_{i+\frac{1}{2}} = Q^{Roe}_{i+\frac{1}{2}} = \max\{\lambda \ |\frac{f(v_{i+1}) - f(v_i)}{v_{i+1} - v_i}|, \quad \epsilon\}$$

. Now this modification removes entropy violating stationary shocks,for example see book of Godlewski and Raviart,page 157,for details.

References

1. Godlewski E. and P.A. Raviart, Hyperbolic systems of conservation laws, mathématiques et Applications, ellipses, Paris, 1991.

2. R.J. LeVeque, Numerical Methods for Conservation Laws, Birkhäuser Verlag, 1992.

3. Smoller J. Shock Waves and Reaction-Diffusion Equations, Grundlehren der mathematischen Wisenschaften 258, Springer-Verlag , New York (Second Edition 1994).